

SL(2,C) Chern-Simons Theory, Flat Connections, and Four-dimensional Quantum Geometry

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ABSTRACT: The present paper analyze SL(2,C) Chern-Simons theory on a class of graph complement 3-manifolds, and its relation with classical and quantum geometries on 4-dimensional manifolds. In classical theory, we explain the correspondence between a class of SL(2,C) flat connections on 3-manifold and the Lorentzian simplicial geometries in 4 dimensions. The class of flat connections on the graph complement 3-manifold is specified by a certain boundary condition. The corresponding simplicial 4-dimensional geometries are made by constant curvature 4-simplices. The quantization of 4d simplicial geometry can be carried out via the quantization of flat connection on 3-manifold in Chern-Simons theory. In quantum SL(2,C) Chern-Simons theory, a basis of physical wave functions is the class of (holomorphic) 3d block, defined by analytically continued Chern-Simons path integral over Lefschetz thimbles. Here we propose that the (holomorphic) 3d block with the proper boundary condition imposed gives the quantization of simplicial 4-dimensional geometry. Interestingly in the semiclassical asymptotic expansion of (holomorphic) 3d block, the leading contribution gives the classical action of simplicial Einstein-Hilbert gravity in 4 dimensions, i.e. Lorentzian 4d Regge action on constant curvature 4-simplices with a cosmological constant. Such a result suggests a relation between SL(2,C) Chern-Simons theory on a class of 3-manifolds and simplicial quantum gravity on 4-dimensional manifolds. This paper presents the details for the results reported in [1].

KEYWORDS: Chern-Simons Theory, Models of Quantum Gravity

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1 Introduction and Overview

Chern-Simons theory in 3-dimensions is the quintessential topological quantum field theory (TQFT) and has been studied extensively since the 1980's (see e.g. [2])¹. Chern-Simons theory with compact gauge group

¹In addition to its importance in the formulation of TQFT [3], Chern-Simons theory has applications in many branches of modern mathematics and physics. The celebrated work of Witten [4], exposed the remarkable relation between Chern-Simons theory with compact gauge group and knot theory. Chern-Simons theory plays an important role in the formulation of the Volume Conjecture,

and its quantization are well understood after the intensive investigations of the last 20 years. However, quantum Chern-Simons theory with complex gauge group $G_{\mathbb{C}}$ (G being a compact Lie group) is a rather open subject. The Chern-Simons theories with gauge groups $G_{\mathbb{C}}$ are noncompact and hence qualitatively different from Chern-Simons theory with a compact group. In general the Hilbert space of Chern-Simons theory with a noncompact group is infinite-dimensional [8, 22–24], while the Hilbert space of the theory with a compact group is finite-dimensional. Recently, there has been substantial progress in the understanding of the theory with complex gauge group [6–8, 24, 25].

This paper focuses on Chern-Simons theory with the complex gauge group $\mathrm{SL}(2, \mathbb{C})$ on a compact oriented 3-manifold \mathcal{M}_3 :

$$CS[\mathcal{M}_3 | A, \bar{A}] = \frac{t}{8\pi} \int_{\mathcal{M}_3} \mathrm{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) + \frac{\bar{t}}{8\pi} \int_{\mathcal{M}_3} \mathrm{tr} \left(\bar{A} \wedge d\bar{A} + \frac{2}{3} \bar{A} \wedge \bar{A} \wedge \bar{A} \right). \quad (1.1)$$

possibly including boundary terms when \mathcal{M}_3 has a boundary. Here $t = k + is$ is the Chern-Simons coupling with $k, s \in \mathbb{R}$, and \bar{t} is assumed to be the complex conjugate of t . The connection 1-form is $A = A^j \tau_j$ in terms of the generators $\tau_j = -\frac{i}{2} \sigma_j$ ($\sigma_{1,2,3}$ denote the Pauli matrices) and takes values in the complex Lie algebra $\mathfrak{sl}_2 \mathbb{C}$. In this work, we focus on a certain class of 3-manifolds \mathcal{M}_3 , the simplest example of which is the graph complement 3-manifold $\mathcal{M}_3 = S^3 \setminus \Gamma_5$, where Γ_5 is the graph with five 4-valent vertices and the single essential crossing depicted in Fig. 1. For a graph embedded in S^3 , the graph complement 3-manifold is obtained by removing the graph as well as the interior of its tubular neighborhood from S^3 . The boundary of $S^3 \setminus \Gamma_5$ is a genus-6 closed 2-surface, which we denote Σ_6 .

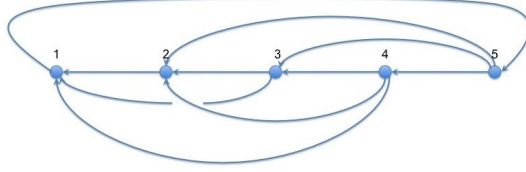


Figure 1. The Γ_5 graph can be drawn with five 4-valent vertices, ten edges ℓ_{ab} , and the curve ℓ_{24} over-crossing ℓ_{13} . It can also be drawn with all vertices being 3-valent, by expanding each 4-valent vertex into two connected 3-valent vertices, which results in 10 vertices and 15 edges. Both ways of drawing Γ_5 lead to the same 3-manifold $S^3 \setminus \Gamma_5$.

Chern-Simons theory with graph defects has been considered in [26] for compact gauge group; and the volume conjecture has been generalized to quantum spin-networks with knotted graphs in [27, 30]. From the mathematical point of view, the space of knotted graphs may be more interesting than the space of knots—due to the fact that the space of trivalent knotted graphs are finitely generated. This means that there is a finite (and small) set of trivalent knotted graphs such that all trivalent knotted graphs can be generated by a few algebraic operations, while the space of knots is a proper subset of the space of trivalent graphs [28]. A recent study of trivalent knotted graphs, from the perspective of perturbative Vassiliev-Kontsevich invariants, gives the right algebraic operations, [29].

Classically, the equations of motion for $\mathrm{SL}(2, \mathbb{C})$ Chern-Simons theory are

$$F = dA + A \wedge A = 0, \quad \bar{F} = d\bar{A} + \bar{A} \wedge \bar{A} = 0, \quad (1.2)$$

that is, the connections A and \bar{A} are flat on the 3-manifold \mathcal{M}_3 . The moduli space of flat connections $\mathcal{M}_{\mathrm{flat}}(\mathcal{M}_3, \mathrm{SL}(2, \mathbb{C}))$ is the space of solutions. When \mathcal{M}_3 has as boundary a closed 2-surface $\Sigma_g = \partial \mathcal{M}_3$, of

which relates knot polynomials to the hyperbolic geometry of 3-manifolds [5–8]. It also has been applied to quantum gravity in 3-dimensions [9, 10]. Moreover, many aspects of String/M-theory and Supersymmetric Gauge Theory have close relationship with Chern-Simons theory (e.g. [11–16]). Chern-Simons theory has also the interesting applications to Loop Quantum Gravity (LQG) in 4 dimensions, in the covariant formulation and black hole physics (e.g. [17–21]).

genus- g , the space of boundary values of $A \in \mathcal{M}_{\text{flat}}(\mathcal{M}_3, \text{SL}(2, \mathbb{C}))$ is a subvariety inside $\mathcal{M}_{\text{flat}}(\Sigma_g, \text{SL}(2, \mathbb{C}))$, the moduli space of $\text{SL}(2, \mathbb{C})$ flat connections on the two-dimensional manifold Σ . In general, $\mathcal{M}_{\text{flat}}(\Sigma_g, \text{SL}(2, \mathbb{C}))$, known as the Hitchin moduli space, is a hyper-Kähler variety of $\dim_{\mathbb{C}} = 6g - 6$, which has 3 distinct complex structures I, J, K [31].² The three corresponding Kähler forms are denoted $\omega_I, \omega_J, \omega_K$. When we consider $\mathcal{M}_{\text{flat}}(\Sigma_g, \text{SL}(2, \mathbb{C}))$ as the phase space of $\text{SL}(2, \mathbb{C})$ Chern-Simons theory, the holomorphic Chern-Simons (Atiyah-Bott-Goldman) symplectic structure ω_{CS} is given by

$$\omega_{CS} = \frac{t}{4\pi} \int_{\Sigma_g} \text{tr} [\delta_1 A \wedge \delta_2 A] = \frac{t}{\pi} [\omega_I - i\omega_K], \quad (1.3)$$

which comes from the holomorphic part of $CS[\mathcal{M}_3 | A, \bar{A}]$. The space of flat connections on \mathcal{M}_3 can be embedded as a subvariety \mathcal{L}_A of $\dim_{\mathbb{C}} = 3g - 3$ in $\mathcal{M}_{\text{flat}}(\Sigma_g, \text{SL}(2, \mathbb{C}))$ by considering the boundary values,

$$\mathcal{L}_A \simeq \mathcal{M}_{\text{flat}}(\mathcal{M}_3, \text{SL}(2, \mathbb{C})). \quad (1.4)$$

The subvariety \mathcal{L}_A is holomorphic with respect to the complex structure J , and is Lagrangian with respect to I and K , i.e. ω_I and ω_K , and hence ω_{CS} , vanish on \mathcal{L}_A [32, 33].

The fact that \mathcal{L}_A is Lagrangian has a clear physical meaning too. Consider an analogy with particle mechanics, which can be seen as a field theory over the time axis. The boundary values are the phase space points at initial and final times, t_0 and t . Introduce a boundary phase space, which is just the Cartesian product of two copies of the phase space one at each time, and has symplectic form $\Omega = dp \wedge dq - dp_0 \wedge dq_0$. The sign on the second term indicates that the initial space is to the past. The statement that the dynamics is a canonical transformation, i.e. that $dp \wedge dq$ is invariant under time evolution, is precisely the statement that the space of orbits of the equations of motion corresponds to a Lagrangian manifold of the boundary phase space. That is, $\Omega|_{\mathcal{L}_D} = 0$, where \mathcal{L}_D is the subset of point of the boundary phase space that are connected by a dynamical orbit. This mechanical analogy was introduced by Tulczyjew precisely with the generalization to field theory in mind [34]. The connections of $\mathcal{M}_{\text{flat}}(\mathcal{M}_3, \text{SL}(2, \mathbb{C}))$ provide dynamical interpolations of the boundary data. So, not only is $\mathcal{M}_{\text{flat}}(\Sigma_g, \text{SL}(2, \mathbb{C}))$ of larger dimension, e.g. there are non-contractible loops in Σ_g that are contractible in \mathcal{M}_3 , but $\mathcal{M}_{\text{flat}}(\mathcal{M}_3, \text{SL}(2, \mathbb{C}))$ is exactly half-dimensional and is Lagrangian.

The complex Fenchel-Nielsen (FN) coordinates $x_m, y_m \in \mathbb{C}$, $m = 1 \cdots 3g - 3$ can be used to locally parametrize the connections of $\mathcal{M}_{\text{flat}}(\Sigma_g, \text{SL}(2, \mathbb{C}))$ [35, 36], using a trinion (or pants) decomposition of the closed 2-surface Σ_g . Here the complex FN “length variable” x_m is the eigenvalue of the holonomy along a closed curve c_m transverse to a tube of the trinion decomposition. The complex FN “twist variable” y_m is the conjugate variable such that ω_{CS} is written as

$$\omega_{CS} = \left(-\frac{t}{2\pi}\right) \sum_{m=1}^{3g-3} \frac{dy_m}{y_m} \wedge \frac{dx_m}{x_m}. \quad (1.5)$$

The explicit relation between y_m and holonomies is given in e.g. [32, 36], and is briefly reviewed in Section 3. In terms of $\{x_m, y_m\}_{m=1}^{3g-3}$, the holomorphic Lagrangian subvariety $\mathcal{L}_A \simeq \mathcal{M}_{\text{flat}}(\mathcal{M}_3, \text{SL}(2, \mathbb{C}))$ can be expressed locally as a set of holomorphic polynomial equations

$$\mathbf{A}_m(x, y) = 0, \quad m = 1, \dots, 3g - 3. \quad (1.6)$$

When \mathcal{M}_3 is the complement of a knot, so that $\partial \mathcal{M}_3 = T^2$, we have $\mathcal{M}_{\text{flat}}(T^2, \text{SL}(2, \mathbb{C})) \simeq \mathbb{C}^* \times \mathbb{C}^* / \mathbb{Z}_2$, and \mathcal{L}_A is the zero-locus of a single holomorphic polynomial $\mathbf{A}(x, y)$, known as the A-polynomial [7, 37].

1.1 Classical Correspondence

Let us focus on the 3-manifold $S^3 \setminus \Gamma_5$ whose boundary is a genus-6 closed surface Σ_6 (see Fig. 2). We are interested in a subspace of $\mathcal{L}_A \simeq \mathcal{M}_{\text{flat}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C}))$, in which the $\text{SL}(2, \mathbb{C})$ flat connections have

²The complex structure I is induced from that of Σ_g , J is from the complex structure of the complex group $\text{SL}(2, \mathbb{C})$, and K is obtained through $K = IJ$.

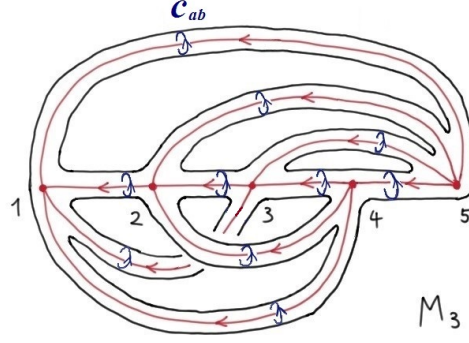


Figure 2. The graph complement 3-manifold M_3 after removing the thickened Γ_5 -graph from S^3 . The 2d boundary $\partial M_3 = \Sigma_6$ of graph complement M_3 is a genus-6 closed 2-surface. A set of meridian closed curves c_{ab} are defined on Σ_6 such that $\Sigma_6 \setminus \{c_{ab}\}$ is a set of 4-holed spheres. $a, b = 1, \dots, 5$ label the vertices of the graph.

4-geometry interpretations. More precisely, a flat connection in the subspace determines the geometry of a convex 4-simplex in 4-dimensional Lorentzian constant curvature spacetime (de-Sitter or Anti-de-Sitter)³. Here we consider 4-dimensional Lorentzian geometry with signature $(-, +, +, +)$. We have that all non-degenerate convex constant curvature 4-simplex geometries with both $\Lambda > 0$ and $\Lambda < 0$ can be described by a class of $\text{SL}(2, \mathbb{C})$ flat connections on the graph complement 3-manifold $S^3 \setminus \Gamma_5$. This can be schematically summarized by

$$\text{A class of } (A, \bar{A}) \text{ in } \mathcal{M}_{\text{flat}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C})) = \text{Constant Curvature 4-Simplex Geometries.} \quad (1.7)$$

The subspace of flat connections on $S^3 \setminus \Gamma_5$ is specified by certain boundary condition imposed on their boundary values on Σ_6 . The boundary condition is introduced in Section 2.2, which can be described briefly in the following way: Σ_6 can be decomposed into five 4-holed spheres $\mathcal{S}_{a=1, \dots, 5}$ by cutting through 10 meridian closed curves in Fig. 2. The boundary condition requires that the boundary value of $A \in \mathcal{M}_{\text{flat}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C}))$ reduces to an $\text{SU}(2)$ flat connection up to gauge transformation, when it is restricted onto one of the 4-holed spheres \mathcal{S}_a . This does not imply that A is an $\text{SU}(2)$ flat connection on all of Σ_6 , since the different 4-holed spheres may correspond to different $\text{SU}(2)$ subgroups in $\text{SL}(2, \mathbb{C})$.

This boundary condition is motivated by the geometrical interpretation of $\text{SU}(2)$ flat connections on a 4-holed sphere \mathcal{S}_a . Each of these connections determine uniquely a convex tetrahedron in constant curvature 3d space (spherical or hyperbolic). The statement holds for a dense subset of $\mathcal{M}_{\text{flat}}(\mathcal{S}_a, \text{SU}(2))$ up to the flat connections corresponding to degenerate geometries. If we consider $\text{PSU}(2)$ flat connections instead of $\text{SU}(2)$, the correspondence becomes 1-to-1 (see Theorem 2.1). This interpretation of $\text{SU}(2)$ flat connections on a 4-holed sphere was introduced in [20] and is reviewed in Section 2.3 (see [38] for a thorough exploration).

A flat connection $A \in \mathcal{M}_{\text{flat}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C}))$ on the Γ_5 graph complement 3-manifold, satisfying the above boundary condition determines uniquely a convex 4-simplex geometry in 4-dimensional Lorentzian spacetime with constant curvature Λ (see Theorem 2.4, see also the analysis in [20]). The closed boundary of the 4-simplex determined by A is formed by 5 constant curvature tetrahedra, which are congruent to the tetrahedral geometries determined by the boundary data of A on the 4-holed spheres \mathcal{S}_a . Again the statement holds up to the flat connections corresponding to degenerate 4-simplex geometries. If we consider $\text{PSL}(2, \mathbb{C})$ flat connections instead of $\text{SL}(2, \mathbb{C})$, the correspondence once again becomes 1-to-1. In the following, we

³A 4-simplex is the elementary building block of the simplicial decomposition of 4-dimensional manifold. It is analogous to the tetrahedron in 3 dimensions and the triangle in 2 dimensions. See Fig. 4 for a projection diagram.

will refer to flat connections satisfying the boundary conditions that put them into correspondence with a 4-simplex geometry as simplicial flat connections.

A simple intuition lies behind the above correspondence between flat connections on a 3-manifold and the geometry of a 4-manifold. The 1-skeleton of a 4-simplex gives a triangulation of the 3-sphere, thought of as the boundary of the 4-simplex. The Γ_5 graph can be viewed as a “dual” graph of the 4-simplex skeleton, in the sense that the fundamental group of $S^3 \setminus \Gamma_5$ is isomorphic to the fundamental group of the 4-simplex skeleton $\pi_1(\text{simplex})$. The isomorphism is unique under a few natural assumptions (see Lemma 2.3). On the one hand, an $\text{SL}(2, \mathbb{C})$ flat connection on $S^3 \setminus \Gamma_5$ is a representation of the fundamental group $\pi_1(S^3 \setminus \Gamma_5)$ up to conjugation. On the other, if the 4-simplex is embedded in a geometrical 4d spacetime $(\mathfrak{M}_4, g_{\alpha\beta})$, the spin connection on \mathfrak{M}_4 gives a representation up to conjugation of $\pi_1(\text{simplex})$ by the holonomies. The isomorphism between $\pi_1(S^3 \setminus \Gamma_5)$ and $\pi_1(\text{simplex})$ identifies the flat connection on $S^3 \setminus \Gamma_5$ and the spin connection on the 4-simplex. More precisely, it identifies the holonomies of the flat connection along the loops in $\pi_1(S^3 \setminus \Gamma_5)$ and the holonomies of spin connection along the closed paths on $\pi_1(\text{simplex})$. In terms of a commutative diagram,

$$\begin{array}{ccc}
\pi_1(S^3 \setminus \Gamma_5) & \xleftarrow{X} & \pi_1(\text{simplex}) \\
\omega_{\text{flat}} \searrow & & \swarrow \omega_{\text{spin}} \\
\langle \{\tilde{H}_{ab} \in \text{SL}(2, \mathbb{C})\}_{a < b} \mid \text{algebraic relations Eqs. (2.12a) – (2.6)} \rangle / \text{conjugation}, & & (1.8)
\end{array}$$

where X denotes the isomorphism between $\pi_1(S^3 \setminus \Gamma_5)$ and $\pi_1(\text{simplex})$ and ω_{flat} and ω_{spin} denote the representations by the flat connection on $S^3 \setminus \Gamma_5$ and the spin connection on \mathfrak{M}_4 , respectively. In this way, the $\text{SL}(2, \mathbb{C})$ flat connections on $S^3 \setminus \Gamma_5$ relate to the spin connections on a spacetime $(\mathfrak{M}_4, g_{\alpha\beta})$. If we take $(\mathfrak{M}_4, g_{\alpha\beta})$ to be a Lorentzian spacetime with constant curvature Λ , and all 10 triangles of the 4-simplex flatly embedded in $(\mathfrak{M}_4, g_{\alpha\beta})$ (i.e. vanishing extrinsic curvature), the holonomy of spin connection along a closed path in $\pi_1(\text{simplex})$ enclosing a single triangle determines the area of the triangle, as well as the embedding property of the triangle, i.e. the 2 normal directions of the triangle embedded in \mathfrak{M}_4 (see Lemma 2.2 and Appendix A). The above relation between ω_{flat} and ω_{spin} , as well as the geometrical properties of the spin connections, result in the correspondence between the $\text{SL}(2, \mathbb{C})$ flat connections on $S^3 \setminus \Gamma_5$ and 4d geometry on a constant curvature 4-simplex.

Given $A \in \mathcal{M}_{\text{flat}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C}))$ corresponding to a constant curvature 4-simplex, it is naturally accompanied by $\tilde{A} \in \mathcal{M}_{\text{flat}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C}))$, which is the complex conjugate of A with respect to the complex structure J induced from the complex group $\text{SL}(2, \mathbb{C})$. The pair A and \tilde{A} determine the same 4-simplex geometry but result in 2 opposite 4d orientations for the 4-simplex. Here we call (A, \tilde{A}) a “parity pair,” because complex conjugation using J naturally relates to a parity inversion in 4d spacetime [20]. This complex conjugation leaves the $\text{SU}(2)$ flat connections invariant, so A and \tilde{A} induce the same $\text{SU}(2)$ flat connections on the 4-holed spheres $\mathcal{S}_{a=1, \dots, 5}$. This is consistent with the fact that the 4-simplex geometries determined by A and \tilde{A} are the same, and give the same set of geometrical tetrahedra on the boundary.

Consider $\mathcal{M}_{\text{flat}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C})) \simeq \mathcal{L}_A$ as a holomorphic Lagrangian subvariety in $\mathcal{M}_{\text{flat}}(\Sigma_6, \text{SL}(2, \mathbb{C}))$. Given $A \in \mathcal{M}_{\text{flat}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C}))$ corresponding to a constant curvature 4-simplex, the complex FN variables of A have direct interpretations in terms of the 4-simplex geometry (see Section 3). The 10 length variables x_{ab} for the closed curves c_{ab} in Fig. 2 relate respectively to the 10 areas a_{ab} of the triangles Δ_{ab} in the 4-simplex. The 10 conjugate twist variable y_{ab} relate respectively to the 10 hyperdihedral angles Θ_{ab} . Each hyperdihedral angle Θ_{ab} between a pair of boundary tetrahedra is hinged by the triangle Δ_{ab} shared by the tetrahedra. Interestingly the canonical conjugacy of a_{ab} and Θ_{ab} that follows from the correspondence between flat connections and their geometrical counterparts, relates to the canonical structure induced by the 4-dimensional Einstein-Hilbert action in General Relativity (GR), see [42] for a derivation in the GR case.

This further motivates the relation between the flat connections on 3-manifolds and (simplicial) gravity on 4-dimensional manifolds.

The phase space of flat connections has complex dimension $\dim_{\mathbb{C}}[\mathcal{M}_{\text{flat}}(\Sigma_6, \text{SL}(2, \mathbb{C}))] = 30$. In addition to the 20 coordinates $\{x_{ab}, y_{ab}\}_{a < b}$, there are 5 pairs of variables $\{x_a, y_a\}_{a=1}^5$ that parametrize the $\text{SU}(2)$ flat connections on $\mathcal{S}_{a=1 \dots 5}$. Geometrically they correspond to the shapes of the 5 constant curvature tetrahedra on the boundary of the geometrical 4-simplex.

1.2 Quantum Correspondence

Our correspondence between $\text{SL}(2, \mathbb{C})$ flat connections on $S^3 \setminus \Gamma_5$ and the constant curvature geometry of 4-simplices inspires a new understanding of 4-dimensional quantum simplicial geometry in terms of the quantization of flat connections on a 3-manifold. For any 3-manifold \mathcal{M}_3 with boundary Σ_g , the quantization of $\mathcal{M}_{\text{flat}}(\Sigma_g, \text{SL}(2, \mathbb{C}))$ with the symplectic structure ω_{CS} results in an operator algebra for the canonically conjugate variables, e.g. the operators for the complex FN variables \hat{x}_m, \hat{y}_m satisfying $\hat{x}_m \hat{y}_m = e^{-\frac{2\pi i \hbar}{i}} \hat{y}_m \hat{x}_m$ ($\hbar \in \mathbb{R}$) and $\hat{x}_m \hat{y}_n = \hat{y}_n \hat{x}_m$ for $n \neq m$. The states are represented as the wave functions $Z(u)$, where u is the logarithmic coordinate $u_m = \ln x_m$. The reader is referred to, e.g. [39, 40], for details of quantizing $\mathcal{M}_{\text{flat}}(\Sigma_g, \text{SL}(2, \mathbb{C}))$. The quantization of the holomorphic Lagrangian subvariety $\mathcal{M}_{\text{flat}}(\mathcal{M}_3, \text{SL}(2, \mathbb{C})) \simeq \mathcal{L}_A$ gives a set of operator constraint equations:

$$\hat{\mathbf{A}}_m(\hat{x}, \hat{y}, \hbar) Z(u) = 0, \quad m = 1, \dots, 3g - 3. \quad (1.9)$$

The solutions $Z(u)$ of the above operator constraint equations are the physical states of $\text{SL}(2, \mathbb{C})$ Chern-Simons theory on \mathcal{M}_3 . A basis of solutions $Z_{CS}^{(\alpha)}(u)$ can be found by the semiclassical WKB method [6, 8, 25]:

$$Z_{CS}^{(\alpha)}(u) = \exp \left[\frac{i}{\hbar} \int_{\mathfrak{C} \subset \mathcal{L}_A}^{u, v^{(\alpha)}} \vartheta + \dots \right] \quad (1.10)$$

The leading term is completely determined by the classical information of phase space and Lagrangian subvariety. Here the Liouville 1-form ϑ satisfies $d\vartheta = \omega_{CS}$ and is integrated along a contour \mathfrak{C} in the Lagrangian subvariety \mathcal{L}_A . The logarithmic coordinates u_m and v_m are related to x_m, y_m by $x_m = e^{u_m}$ and $y_m = e^{-\frac{2\pi}{i} v_m}$. The label α indexes the branches of \mathcal{L}_A on which a unique set of v_m can be solved for from u_m through $\mathbf{A}_m(x, y) = 0$. The end point of the contour \mathfrak{C} , which labels $Z_{CS}^{(\alpha)}(u)$, is a flat connection determined by $u, v^{(\alpha)}$ in $\mathcal{M}_{\text{flat}}(\mathcal{M}_3, \text{SL}(2, \mathbb{C})) \simeq \mathcal{L}_A$ (or more precisely, in the cover space of $\mathcal{M}_{\text{flat}}(\mathcal{M}_3, \text{SL}(2, \mathbb{C})) \simeq \mathcal{L}_A$). Thus $Z_{CS}^{(\alpha)}(u)$ is associated to a single flat connection $A \in \mathcal{M}_{\text{flat}}(\mathcal{M}_3, \text{SL}(2, \mathbb{C}))$. The starting point of \mathfrak{C} is conventional and corresponds to a choice of overall phase for $Z_{CS}^{(\alpha)}(u)$. The ellipsis “...” in Eq. (1.10) stands for the quantum corrections, which in principle can be obtained recursively from the operator constraint equations. The semiclassical wave function $Z_{CS}^{(\alpha)}(u)$, often called an *holomorphic 3d block*, can also be formulated nonperturbatively as a “state-integral model,” see [8, 41].

The holomorphic 3d block $Z_{CS}^{(\alpha)}(u)$ can also be defined by a functional integral of the holomorphic part of $CS[\mathcal{M}_3 | A, \tilde{A}]$ over a certain integration cycle, known as a Lefschetz thimble [6]. The Lefschetz thimble is an integration cycle that only contains a single critical point of the action; this provides another way to understand the association between $Z_{CS}^{(\alpha)}(u)$ and a single flat connection on \mathcal{M}_3 .

The holomorphic 3d block $Z_{CS}^{(\alpha)}(u)$ plays a central role in the quantum part of this work. We again specialize to the 3-manifold $S^3 \setminus \Gamma_5$ with boundary Σ_6 and impose boundary conditions on Σ_6 to pick out the flat connections in $\mathcal{M}_{\text{flat}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C}))$ corresponding to constant curvature 4-simplices. Given such an $A \in \mathcal{M}_{\text{flat}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C}))$, as well as its parity partner \tilde{A} , we can construct an holomorphic 3d block $Z_{CS}^{(\alpha)}(u)$ associated with A and using \tilde{A} as a reference. We simply let A be the end point of the contour \mathfrak{C} and use \tilde{A} as its initial point. Our classical correspondence between flat connection on $S^3 \setminus \Gamma_5$ and constant curvature 4-simplex geometries suggests that the so constructed $Z_{CS}^{(\alpha)}(u)$ is a wave function for the *quantum 4d geometry* of a constant curvature 4-simplex. Schematically,

$$Z_{CS}^{(\alpha)}(u) \text{ with Boundary Condition} = \text{Quantum Constant Curvature 4-Simplex Geometry.} \quad (1.11)$$

This quantum correspondence indicates that the asymptotic expansion of $Z_{CS}^{(\alpha)}(u)$ in Eq.(1.10) should have the classical action for the simplicial 4d geometry as its leading term. In particular, due to the relation between the symplectic structures of flat connections and 4d simplicial gravity, it is natural to expect that the leading term should give the action of 4d gravity in the simplicial context.

This expectation is confirmed by the analysis in Section 4.2. We show that the leading asymptotic behavior of $Z_{CS}^{(\alpha)}(u)$ is a simplicial discretization of the four-dimensional Einstein-Hilbert action on a constant curvature 4-simplex

$$S_{Regge}^\Lambda = \sum_{a<b} a_{ab} \Theta_{ab} - \Lambda \text{Vol}_4^\Lambda, \quad (1.12)$$

we call this the curved Regge action, and it is expressed here up to an integration constant and a term depending on the lift to the logarithmic variables (u, v) . The coefficient Λ is the cosmological constant and can also be identified as the constant curvature of the 4-simplex, while Vol_4^Λ is its 4-volume. We refer the reader to, e.g. [43–46], for the derivation of the curved 4d Regge action through a discretization of the Einstein-Hilbert action (see also [20] for a summary).

Because $Z_{CS}^{(\alpha)}(u)$ is holomorphic, its leading asymptotic behavior is not necessarily an oscillatory phase. If we consider the full $\text{SL}(2, \mathbb{C})$ Chern-Simons action $CS[\mathcal{M}_3 | A, \bar{A}]$ including both holomorphic and anti-holomorphic parts, we are interested in the 3d block $Z_{CS}^{(\alpha)}(u)Z_{CS}^{(\bar{\alpha})}(\bar{u})$, where $Z_{CS}^{(\bar{\alpha})}(\bar{u})$ is associated to \bar{A} . For a flat connection with a corresponding 4-simplex geometry, the leading asymptotic behavior of $Z_{CS}^{(\alpha)}(u)Z_{CS}^{(\bar{\alpha})}(\bar{u})$ is an oscillatory phase:

$$\begin{aligned} & Z_{CS}^{(\alpha)}(u) Z_{CS}^{(\bar{\alpha})}(\bar{u}) \\ &= \exp \left[\frac{i}{\hbar} 2\text{Re} \left(\frac{\Lambda t}{12\pi i} \right) S_{Regge}^\Lambda + \frac{i}{\hbar} 2\text{Re} \left(\frac{\Lambda t}{12} \right) \sum_{a<b} \mathbf{K}_{ab} \mathbf{a}_{ab} + iC_{\text{int}} + \dots \right]. \end{aligned} \quad (1.13)$$

This is shown in section 4. Thus we see that $Z_{CS}^{(\alpha)}(u)Z_{CS}^{(\bar{\alpha})}(\bar{u})$ is an analog of the functional integral quantization of the Einstein-Hilbert action in the simplicial context,

$$\exp \left[\frac{i}{\ell_P^2} \int_{\mathcal{M}_4} R + \text{“Quantum Corrections”} \right]. \quad (1.14)$$

With this analogy in mind, we identify the gravitational constant G_N in terms of Chern-Simons coupling t and cosmological constant Λ as

$$G_N = \left| \frac{3}{2\text{Im}(t)\Lambda} \right|. \quad (1.15)$$

The quantity $C_{\text{int}} \in \mathbb{R}$ in (1.15) is an (integration) constant that is independent of the 4-simplex geometry. The additional term $\frac{i}{\hbar} 2\text{Re} \left(\frac{\Lambda t}{12} \right) \sum_{a<b} \mathbf{K}_{ab} \mathbf{a}_{ab}$ ($\mathbf{K}_{ab} \in \mathbb{Z}$) in the leading asymptotics comes from the choice of lift of the FN variables x_m and y_m to the logarithmic variables u_m and v_m . This term disappears trivially when $t \in i\mathbb{R}$. However, for general complex t , the additional term can also be made to disappear by imposing a quantization condition on the triangle areas \mathbf{a}_{ab} :

$$2\text{Re} \left(\frac{\Lambda t}{12} \right) \sum_{a<b} \mathbf{K}_{ab} \delta \mathbf{a}_{ab} \in 2\pi\hbar\mathbb{Z}. \quad (1.16)$$

Indeed, this quantization condition is natural: When the boundary condition on $A \in \mathcal{M}_{\text{flat}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C}))$ is imposed by the Wilson graph operator, the quantization condition is automatically satisfied (see Section 5). The quantization condition is also consistent with the discrete area spectrum in Loop Quantum Gravity (LQG) [47].

The main part of the paper discusses the correspondences at the single 4-simplex level, this being the most crucial step in building the model for more general situations. The analysis is generalized, in

Section 7, to a simplicial complex with an arbitrary number of 4-simplices. In Section 7, a class of 3-manifolds \mathcal{M}_3 are constructed by gluing several graph complements $S^3 \setminus \Gamma_5$. In this section flat connections $A \in \mathcal{M}_{\text{flat}}(\mathcal{M}_3, \text{SL}(2, \mathbb{C}))$ are identified that determine all possible (nondegenerate) 4-dimensional simplicial geometries on a simplicial complex \mathcal{K}_4 . In each resulting simplicial geometry, the 4-simplices are of constant curvature Λ , while the large simplicial geometry built by many 4-simplices can approximate arbitrary smooth geometry on a 4-manifold. However a generic $A \in \mathcal{M}_{\text{flat}}(\mathcal{M}_3, \text{SL}(2, \mathbb{C}))$ having 4d simplicial geometry correspondence may result in a non-uniform 4d orientation on the simplicial complex, i.e. different 4-simplex may obtain different 4d orientations. For an orientable simplicial complex, we can find the class of flat connections on \mathcal{M}_3 , which not only determine all possible (nondegenerate) 4-dimensional simplicial geometries on \mathcal{K}_4 , but also induce the global 4d orientations. Each flat connection A in the class is accompanied by its *global parity* partner \tilde{A} . We construct the Chern-Simons 3d block $Z_{CS}^{(\alpha)}(\mathcal{M}_3|u) Z_{CS}^{(\bar{\alpha})}(\mathcal{M}_3|\bar{u})$ that associates with A and with reference \tilde{A} in the same way as above. The asymptotic expansion in \hbar of the resulting 3d block generalize Eq.(1.13) to the level of simplicial complex:

$$\begin{aligned} & Z_{CS}^{(\alpha)}(\mathcal{M}_3|u) Z_{CS}^{(\bar{\alpha})}(\mathcal{M}_3|\bar{u}) \\ &= \exp \left[\frac{i}{\hbar} 2\text{Re} \left(\frac{\Lambda t}{12\pi i} \right) \mathbf{S}_{\text{Regge}}^\Lambda + \frac{i}{\hbar} 2\text{Re} \left(\frac{\Lambda t}{12} \right) \sum_{\Delta} \mathbf{K}_{\Delta} \mathbf{a}_{\Delta} + i\mathbf{C}_{\text{int}} + \dots \right]. \end{aligned} \quad (1.17)$$

where $\mathbf{S}_{\text{Regge}}^\Lambda$ is the 4-dimensional Lorentzian Regge action on the entire simplicial complex \mathcal{K}_4 :

$$\mathbf{S}_{\text{Regge}}^\Lambda = \sum_{\Delta \text{ internal}} \mathbf{a}(\Delta) \varepsilon(\Delta) + \sum_{\Delta \text{ boundary}} \mathbf{a}(\Delta) \Theta(\Delta) - \Lambda \sum_{\sigma} \text{Vol}_4^\Lambda(\sigma) \quad (1.18)$$

Here Δ denotes a triangle in \mathcal{K}_4 and σ denotes a 4-simplex. If we denote $\Theta_{\Delta}(\sigma)$ is the hyperdihedral boost angle of Δ in the 4-simplex σ (the same as Θ_{ab} above), $\varepsilon(\Delta)$ is the Lorentzian deficit angle defined by $\varepsilon(\Delta) := \sum_{\sigma, \Delta \subset \sigma} \Theta_{\Delta}(\sigma)$ for internal Δ , and $\Theta(\Delta)$ is the Lorentzian boundary hyperdihedral angle defined by $\Theta(\Delta) := \sum_{\sigma, \Delta \subset \sigma} \Theta_{\Delta}(\sigma)$ for boundary Δ . In Eq.(1.17) the additional term $\frac{i}{\hbar} 2\text{Re} \left(\frac{\Lambda t}{12} \right) \sum_{\Delta} \mathbf{K}_{\Delta} \mathbf{a}_{\Delta}$ ($\mathbf{K}_{\Delta} \in \mathbb{Z}$) is again disappear when $t \in i\mathbb{R}$, or when the above quantization condition is satisfied for general t .

The above asymptotics expansion in \hbar suggests that the Chern-Simons 3d block $Z_{CS}^{(\alpha)}(u) Z_{CS}^{(\bar{\alpha})}(\bar{u})$, that associates with a flat connection on \mathcal{M}_3 corresponding to 4d simplicial geometry on \mathcal{K}_4 , is a wave function of 4-dimensional simplicial quantum gravity, where its subleading terms in \hbar gives the quantum correction to the classical Einstein-Hilbert action.

1.3 Wilson Graph Operator and Loop Quantum Gravity

The analysis in the present paper is a continuation of the work done in [20], where a class of Wilson graph operators are studied in $\text{SL}(2, \mathbb{C})$ Chern-Simons theory on S^3 . The Wilson graph operators are defined by Γ_5 graph embedded in S^3 colored by certain principle unitary irreducible representations of $\text{SL}(2, \mathbb{C})$. The definition is summarized in Section 5. In [20], we have studied the Chern-Simons expectation value \mathcal{A} of the Wilson graph operators on S^3 , and in particular the asymptotic behavior of \mathcal{A} in the “double-scaling limit”, i.e. scaling both Chern-Simons coupling t and Wilson-graph representation labels to infinity, and keeping their ratio fixed. Under the double-scaling limit, the Chern-Simons expectation value \mathcal{A} of the Wilson graph operator produces 4d Regge action S_{Regge}^Λ of a constant curvature 4-simplex as the leading asymptotics:

$$\mathcal{A} = e^{\frac{i}{\ell_P^2} S_{\text{Regge}}^\Lambda + \dots} + e^{-\frac{i}{\ell_P^2} S_{\text{Regge}}^\Lambda + \dots} \quad (1.19)$$

up to an possible overall phase factor. Here \dots stands for the subleading terms in the double-scaling limit. In the sense of asymptotics and relation with simplicial gravity, \mathcal{A} can be viewed as a 4d gravity analog of quantum 6j symbol in Turaev-Viro model of 3d quantum gravity [57, 58]⁴.

⁴The double-scaling limit of quantum 6j symbol gives the 3d Regge action on a constant curvature tetrahedron [58].

As a motivation for the analysis in [20], the Chern-Simons expectation value \mathcal{A} has a close relationship with Loop Quantum Gravity (LQG). LQG is an attempt to make a background independent, non-perturbative quantization of 4-dimensional gravity, for reviews, see [48–50]. *Spinfoam amplitude* is one of the central object for the covariant dynamics of LQG, which adapts the idea of path integral quantization to the framework of LQG. A spinfoam amplitude is defined on a 4-dimensional simplicial complex \mathcal{K}_4 and encodes the transition amplitude for the boundary quantum 3-geometry. In LQG, the quantum 3-dimensional geometry is described by *spin-network states*. The spinfoam amplitude sums over the history of spin-networks, and suggests a foam-like quantum spacetime structure. An important building block of the spinfoam amplitude is Engle-Pereira-Rovelli-Livine (EPRL) 4-simplex amplitude \mathcal{A}_{EPRL} associated to a 4-simplex σ in \mathcal{K}_4 [51]⁵. The Chern-Simons expectation value \mathcal{A} of the Wilson graph is an deformation of EPRL 4-simplex amplitude, in the sense that \mathcal{A} approaches \mathcal{A}_{EPRL} asymptotically in the decoupling limit, i.e. Chern-Simons coupling t is scaled to infinity, keeping Wilson graph representations fixed (see [20] or Section 6). Such an deformation has been motivated from mainly two perspectives: (1) the studies of relation between LQG and Topological Quantum Field Theory (TQFT) [17, 18, 53], (2) the quantum group deformation of spinfoam amplitude and inclusion of cosmological constant [19, 54]. We have the following commutative diagram for the relations among \mathcal{A} , \mathcal{A}_{EPRL} , and Regge action S_{Regge}^Λ or S_{Regge} with or without cosmological constant term:

$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\text{double-scaling limit}} & e^{\frac{i}{\ell_P^2} S_{Regge}^\Lambda} + e^{-\frac{i}{\ell_P^2} S_{Regge}^\Lambda} \\
\downarrow \text{decoupling} & & \downarrow \Lambda \rightarrow 0 \\
\mathcal{A}_{EPRL} & \xrightarrow{\text{large-}j \text{ limit}} & e^{\frac{i}{\ell_P^2} S_{Regge}} + e^{-\frac{i}{\ell_P^2} S_{Regge}}
\end{array} \tag{1.20}$$

The asymptotic behavior downstairs is shown for EPRL 4-simplex amplitude \mathcal{A}_{EPRL} in [55, 56]. S_{Regge} from the asymptotics of \mathcal{A}_{EPRL} is the Regge action without cosmological constant for a *flat* 4-simplex, while S_{Regge}^Λ from the Chern-Simons expectation value \mathcal{A} is the Regge action with cosmological constant Λ for a *constant curvature* 4-simplex, the same as Eq.(1.12). Therefore \mathcal{A} is a deformation of \mathcal{A}_{EPRL} in the spinfoam amplitude, which includes the cosmological constant in the framework of LQG.

Probably one has noticed that the 4-dimensional Lorentzian Regge action S_{Regge}^Λ appears in the leading asymptotics of both Chern-Simons expectation value \mathcal{A} of Wilson graph operator and Chern-Simons 3d block $Z_{CS}^{(a)}(u) Z_{CS}^{(\bar{a})}(\bar{u})$. It is not a coincidence as suggested in Section 5. Firstly it turns out that the double-scaling limit in Chern-Simons theory on S^3 with Wilson graph is the same as the semiclassical limit $\hbar \rightarrow 0$ of Chern-Simons theory on the graph complement, keeping the boundary data fixed. Secondly the Chern-Simons expectation value \mathcal{A} can be understood as an inner product

$$\mathcal{A} = \langle N(\Gamma_5) | S^3 \setminus \Gamma_5 \rangle \tag{1.21}$$

where $|N(\Gamma_5)\rangle$ is the Chern-Simons state on the tubular neighborhood of Γ_5 excited by the Wilson graph operator, and $|S^3 \setminus \Gamma_5\rangle$ is the Chern-Simons ground state on $S^3 \setminus \Gamma_5$. In the double-scaling limit, the Wilson graph operators in [20] defining \mathcal{A} impose the right boundary condition to the boundary Σ_6 of $S^3 \setminus \Gamma_5$ (including the quantization condition Eq.(1.16)), so that the boundary condition picks the parity pair of flat connections A, \tilde{A} on $S^3 \setminus \Gamma_5$, which determines a constant curvature 4-simplex geometry. In other words, the state $|N(\Gamma_5)\rangle$ is a “semiclassical state” picked at the right phase space point in $\mathcal{M}_{\text{flat}}(\Sigma_6, \text{SL}(2, \mathbb{C}))$. The state $|S^3 \setminus \Gamma_5\rangle$ is a linear combination of Chern-Simons 3d blocks $Z_{CS}^{(a)}(u) Z_{CS}^{(\bar{a})}(\bar{u})$ on $S^3 \setminus \Gamma_5$. The peakedness of $|N(\Gamma_5)\rangle$ selects the right pair of 3d blocks that associates with A and \tilde{A} respectively, which have respectively $e^{\frac{i}{\ell_P^2} S_{Regge}^\Lambda}$ and $e^{-\frac{i}{\ell_P^2} S_{Regge}^\Lambda}$ in their leading asymptotics. Note that in the study of Chern-Simons 3d block, we only

⁵It is sometimes also called EPRL/FK, for Freidel and Krasnov, when referring to the version for Euclidean gravity [52]

obtain a single phase because the 3d block is defined to associate with A and with reference \tilde{A} , which is the ratio of two 3d blocks with A and \tilde{A} respectively (their overall phase ambiguities cancel).

The separate studies of Chern-Simons 3d block and Wilson graph operator clarify their different roles played in the asymptotics of \mathcal{A} . The Regge-action asymptotic behavior of \mathcal{A} crucially depends on the peakedness of $|N(\Gamma_5)\rangle$ created by the Wilson graph operator. However different Wilson graph operators may produce the same peakedness in phase space⁶, thus leads to the same asymptotics of \mathcal{A} . The close relationship with EPRL 4-simplex amplitude motivates us to study the particular type of Wilson graph operators in [20]. But in principle other types of Wilson graph operators are possible to work equally well, when they produce the same peakedness⁷. However independent of the choice of Wilson graphs, the essential role played behind the Regge-action asymptotics of \mathcal{A} is Chern-Simons 3d block on $S^3 \setminus \Gamma_5$ with the right boundary condition imposed. Therefore the Chern-Simons 3d block $Z_{CS}^{(a)}(u) Z_{CS}^{(\bar{a})}(\bar{u})$ studied here plays an important role in the covariant formulation of LQG. The classical and quantum correspondences between flat connection on 3-manifold and simplicial geometry on 4-manifold studied here may be viewed as a re-formulation of covariant LQG, by emphasizing the relation with $SL(2, \mathbb{C})$ Chern-Simons theory.

Our result for the quantum correspondence suggests the Chern-Simons 3d block $Z_{CS}^{(a)}(u) Z_{CS}^{(\bar{a})}(\bar{u})$ to be the wave function of simplicial quantum gravity in 4 dimensions. Given its relation with LQG, $Z_{CS}^{(a)}(u) Z_{CS}^{(\bar{a})}(\bar{u})$ may be understood as the physical wave function for LQG in 4 dimensions, at least in the level of simplicial geometry. As a future research, it is interesting to find the behavior of $Z_{CS}^{(a)}(u) Z_{CS}^{(\bar{a})}(\bar{u})$ under the procedure which corresponds to refining the simplicial complex \mathcal{K}_4 . It should shed light on the understanding of continuum limit in the covariant formulation of LQG.

The physical wave function of LQG should describe the quantum transitions in a 4-dimensional region, between the boundary quantum 3d geometries. In this logic, the boundary data of $Z_{CS}^{(a)}(u) Z_{CS}^{(\bar{a})}(\bar{u})$, being the flat connections on the 2d boundary of the graph complement 3-manifold, should describe the quantum 3d geometry in LQG. Indeed, as it is discussed in Section 6, the boundary data of $Z_{CS}^{(a)}(u) Z_{CS}^{(\bar{a})}(\bar{u})$ relate naturally the spin-network states, which quantize 3d geometry in the kinematical framework of LQG.

1.4 Structure of the Paper

The structure of the present paper is summarized as follows: Section 2 explains the classical correspondence between the $SL(2, \mathbb{C})$ flat connections on $S^3 \setminus \Gamma_5$ specified by boundary conditions and constant curvature 4-simplex geometries in 4 dimensions. Based on the correspondence, Section 3 identifies the complex Fenchel-Nielsen length and twist variables to the triangle areas and hyperdihedral angles in constant curvature 4-simplex geometry. Section 4 discuss the correspondence between quantum $SL(2, \mathbb{C})$ Chern-Simons theory on $S^3 \setminus \Gamma_5$ and quantum 4-simplex geometry. After a brief review of quantum Chern-Simons theory and holomorphic 3d block in Section 4.1, we analyze in Section 4.2 the asymptotic expansion of Chern-Simons 3d block, whose leading order gives 4-dimensional Regge action with cosmological constant on constant curvature 4-simplex. Section 5 discusses the relation with [20], in which the Wilson graph operators is used to impose the right boundary condition. Section 6 discuss the relation between $SL(2, \mathbb{C})$ Chern-Simons theory and Loop Quantum Gravity in 4 dimensions. In Section 7, the correspondence is generalized from a single 4-simplex to 4d simplicial complex. A class of Chern-Simons 3d block is defined to produce 4d Regge action on simplicial complex in the asymptotics.

2 From Flat Connections on a 3-Manifold to 4d Simplicial Geometry

2.1 Flat Connections on a Graph-Complement 3-Manifold

Consider the pentagon graph Γ_5 of figure 1 to be embedded in a 3-sphere S^3 , and let $N(\Gamma_5)$ be (the interior of) its tubular neighborhood in such a manifold. Hence, define the 3-manifold $M_3 := S^3 \setminus N(\Gamma_5)$. Notice that

⁶For instance, for harmonic oscillator, different squeezed coherent state can have the same peakedness.

⁷The different types of Wilson graphs having the same peakedness may relate to the spinfoam amplitudes defined in [59].

$\partial M_3 = \overline{\partial N(\Gamma_5)}$. By a slight abuse of notation we will often write

$$M_3 = S^3 \setminus \Gamma_5. \quad (2.1)$$

The moduli space of flat $\mathfrak{sl}_2\mathbb{C}$ connections on M_3 is defined as

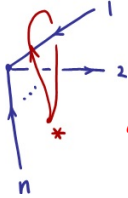
$$\mathcal{M}_{\text{flat}}(M_3, \text{SL}(2, \mathbb{C})) = \text{Hom}(\pi_1(M_3), \text{SL}(2, \mathbb{C})) / \text{conjugation}, \quad (2.2)$$

i.e. as the space of representations ρ of the fundamental group of M_3 in the group $\text{SL}(2, \mathbb{C})$, up to conjugation.

Defined as above, the moduli space of flat connections is often badly behaved, e.g. it is non-Hausdorff. It is customary — and enough to our purposes — to make a further restriction to the so-called ‘character variety’, which is an algebraic variety [60].

The fundamental group $\pi_1(M)$ of a graph complement M is easily characterized via a generalized Wirtinger presentation [61]. This goes as follows: (i) project the graph onto a plane; (ii) take a point $*$ in front of the plane as the base point; (iii) take as generators of $\pi_1(M)$ the independent loops starting and ending at $*$, each going around one single edge only once and crossing the plane of the projected graph only twice; (iv) notice that every crossing breaks the original underlying edge into two pieces, whose associated loops $l^{(1)}, l^{(2)}$ must be considered as independent generators. The so obtained generators must satisfy the following two sets of relations:

- if n edges meet at a vertex (all supposed to be ingoing for the moment), then



$l_n \cdots l_2 l_1 = e,$

and nexteqnotsurewhatnexteqis

(2.3)

where e denotes the identity in $\pi_1(M)$, and we have supposed them to be numbered from 1 to n in a clockwise fashion on the projection plane. To change the i -th edge from ingoing to outgoing, substitute l_i with l_i^{-1} ;

- if under projection onto the plane, an edge with generator \tilde{l} over-crosses another edge, the latter gets associated with two independent generators $l^{(1)}$ and $l^{(2)}$ (as in the figure below). Then the three generators \tilde{l} , $l^{(1)}$ and $l^{(2)}$ satisfy

$l^{(1)} = \tilde{l} l^{(2)} \tilde{l}^{-1},$



(2.4)

In this way, $\pi_1(M_3 = S^3 \setminus \Gamma_5)$ can be computed straightforwardly. To fix the notation, label the vertices of Γ_5 as in figure ?? with an index $a \in \{1, \dots, 5\}$, and thus name its (unoriented) edges $\ell_{ab} = \ell_{ba}$. The generators of $\pi_1(M_3)$ are then the loops l_{ab} associated to every edge ℓ_{ab} of Γ_5 but ℓ_{13} , which is broken by a crossing, and to which one has hence to associate two distinct generators $l_{13}^{(1)}$ and $l_{13}^{(2)}$. A representation $\rho \in \text{Hom}(\pi_1(M_3), \text{SL}(2, \mathbb{C}))$ maps each of the previous generators to an element of $\text{SL}(2, \mathbb{C})$, i.e. $\rho(l_{ab}) = \tilde{H}_{ab}$ for every $(ab) \neq (13)$ and $\rho(l_{13}^{(i)}) = \tilde{H}_{13}^{(i)}$, for $i \in \{1, 2\}$. As a consequence of equations 2.3 and 2.4, these group elements (holonomies) must satisfy:

$$\text{vertex 1 : } \tilde{H}_{14} \tilde{H}_{13}^{(1)} \tilde{H}_{12} \tilde{H}_{15} = 1, \quad (2.5a)$$

$$\text{vertex 2 : } \tilde{H}_{12}^{-1} \tilde{H}_{24} \tilde{H}_{23} \tilde{H}_{25} = 1, \quad (2.5b)$$

$$\text{vertex 3 : } \tilde{H}_{23}^{-1} (\tilde{H}_{13}^{(2)})^{-1} \tilde{H}_{34} \tilde{H}_{35} = 1, \quad (2.5c)$$

$$\text{vertex 4 : } \tilde{H}_{34}^{-1} \tilde{H}_{24} \tilde{H}_{14}^{-1} \tilde{H}_{45} = 1, \quad (2.5d)$$

$$\text{vertex 5 : } \tilde{H}_{25}^{-1} \tilde{H}_{35}^{-1} \tilde{H}_{45}^{-1} \tilde{H}_{15}^{-1} = 1, \quad (2.5e)$$

$$\text{crossing} : \tilde{H}_{13}^{(1)} = \tilde{H}_{24} \tilde{H}_{13}^{(2)} \tilde{H}_{24}^{-1}. \quad (2.6)$$

Notice that all the above holonomies, collectively referred to as $\{\tilde{H}_{ab}\}$, have the same base-point $*$ $\in S^3 \setminus \Gamma_5$.

The moduli space $\mathcal{M}_{\text{flat}}(M_3, \text{SL}(2, \mathbb{C}))$ is defined as the $\{\tilde{H}_{ab}\}$ modulo simultaneous conjugation by a $g \in \text{SL}(2, \mathbb{C})$, i.e. $\{\tilde{H}_{ab}\} \sim \{g \tilde{H}_{ab} g^{-1}\}$.

2.2 Boundary Conditions

For our geometrical purposes, we are not interested in a generic connection in $\mathcal{M}_{\text{flat}}(M_3, \text{SL}(2, \mathbb{C}))$. Rather, we want to restrict to connections satisfying a certain type of boundary conditions at

$$\Sigma_6 := \partial M_3 = \partial \overline{N(\Gamma_5)}, \quad (2.7)$$

which is a closed 2-surface of genus 6.

The restriction of a connection $A \in \mathcal{M}_{\text{flat}}(M_3, \text{SL}(2, \mathbb{C}))$ to the boundary surface Σ_6 gives an element of $\mathcal{M}_{\text{flat}}(\Sigma_6, \text{SL}(2, \mathbb{C}))$. In this sense one can write

$$\mathcal{M}_{\text{flat}}(M_3, \text{SL}(2, \mathbb{C})) \subset \mathcal{M}_{\text{flat}}(\Sigma_6, \text{SL}(2, \mathbb{C})). \quad (2.8)$$

On Σ_6 , we specify 10 meridian curves $\{c_{ab}\}$ each cutting one edge of Γ_5 transversally. Hence,

$$\Sigma_6 \setminus \{c_{ab}\} = \bigcup_a \mathcal{S}_a \quad (2.9)$$

where $\mathcal{S}_a \cong S^3 \setminus \{4\text{pts}\}$ is the four-punctured sphere associated to the a -th vertex of Γ_5 . A representation $\sigma \in \text{Hom}(\pi_1(\Sigma_6), \text{SL}(2, \mathbb{C}))$ when restricted to \mathcal{S}_a gives a representation $\sigma|_{\mathcal{S}_a} \in \text{Hom}(\pi_1(\mathcal{S}_a), \text{SL}(2, \mathbb{C}))$ (defined up to global $\text{SL}(2, \mathbb{C})$ conjugation).

The way we think of these punctured spheres is as (the boundary of) a tetrahedron whose ‘quanta of area’ are ‘concentrated’ at the punctures in form of defects. We want each of these tetrahedra to define a three-dimensional space-like frame in (A)dS.

With this geometrical picture in mind we define the following boundary conditions: a representation $\sigma \in \text{Hom}(\pi_1(\Sigma_6), \text{SL}(2, \mathbb{C}))$ is said to satisfy the *geometric boundary conditions* if there exists five elements $g_a \in \text{SL}(2, \mathbb{C})$, such that

$$g_a(\sigma|_{\mathcal{S}_a})g_a^{-1} \in \text{Hom}(\pi_1(\mathcal{S}_a), \text{SU}(2)). \quad (2.10)$$

In words, an $\text{SL}(2, \mathbb{C})$ representation of the fundamental group of Σ_6 is said to satisfy the geometric boundary conditions if on each four-punctured sphere \mathcal{S}_a it restricts to an $\text{SU}(2)$ representation up to a conjugation by an element $g_a \in \text{SL}(2, \mathbb{C})$:

$$\forall a \exists g_a \in \text{SL}(2, \mathbb{C}) \quad \text{such that} \quad g_a \tilde{H}_{ab} g_a^{-1} =: H_b(a) \in \text{SU}(2) \quad \forall b, b \neq a \quad (2.11)$$

We call the gauge associated to such a set of $\{g_a\}$, the ‘time gauge’.

An immediate consequence of the geometric boundary conditions is that equations 2.5 can be written after conjugation by $g_a \in \text{SL}(2, \mathbb{C})$ as equations in $\text{SU}(2)$:

$$\text{vertex 1 : } H_4(1)H_3(1)H_2(1)H_5(1) = 1, \quad (2.12a)$$

$$\text{vertex 2 : } H_1^{-1}(2)H_4(2)H_3(2)H_5(2) = 1, \quad (2.12b)$$

$$\text{vertex 3 : } H_2^{-1}(3)H_1^{-1}(3)H_4(3)H_5(3) = 1, \quad (2.12c)$$

$$\text{vertex 4 : } H_3^{-1}(4)H_2^{-1}(4)H_1^{-1}(4)H_5(4) = 1, \quad (2.12d)$$

$$\text{vertex 5 : } H_1^{-1}(5)H_2^{-1}(5)H_3^{-1}(5)H_4^{-1}(5) = 1. \quad (2.12e)$$

We will refer to these equations as the ‘closure equations’.

The missing information with respect to equations 2.5 and 2.6 can be recast in terms of some $G_{ab} \in \text{SL}(2, \mathbb{C})$ defined as

$$G_{ba} := g_b^{-1} g_a \quad \text{for all } (ab), \text{ except } G_{13} := g_1^{-1} [g_2 H_4(2) g_2^{-1}] g_3. \quad (2.13)$$

Then such information takes the form of ‘parallel transport equations’ encoding $\tilde{H}_{ab} = \tilde{H}_{ba}$

$$G_{ab} H_b(a) G_{ba} = H_a(b), \quad (2.14)$$

and of ‘bulk equations’ encoding the positions of the crossing

$$G_{ac} G_{cb} G_{ba} = 1 \quad (abc) \in \{125, 235, 345, 124, 234\} \quad (2.15a)$$

$$G_{13} G_{32} G_{21} = H_4(2) \quad (2.15b)$$

We name the set of connections satisfying the geometric boundary condition

$$\mathcal{M}_{\text{flat}}^{\text{BC}}(\Sigma_6, \text{SL}(2, \mathbb{C})) \subset \mathcal{M}_{\text{flat}}(\Sigma_6, \text{SL}(2, \mathbb{C})) \quad (2.16)$$

In section ??, we will come back to these boundary conditions and express them in terms of a set of preferred coordinates, the complex Fenchel-Nielsen coordinates. These are Darboux coordinates on $\mathcal{M}_{\text{flat}}(\Sigma_6, \text{SL}(2, \mathbb{C}))$ with respect to the canonical Atiyah-Bott-Goldman symplectic structure induced by the Chern-Simons theory.

2.3 Geometrical Interpretation of Boundary Condition: Curved Tetrahedron

Henceforth, when there is room for confusion, we will denote embedded geometric quantities (in either S^3 , H^3 , dS , or AdS) by gothic-face quantities, to be distinguished, e.g., from abstract vectors and bivectors. It is the aim of our reconstruction theorems to state correspondences among these two—embedded versus abstract—sets of quantities.

As we have already anticipated, there is a precise correspondence between $SU(2)$ flat connections on a four-holed sphere and tetrahedral geometries flatly embedded in S^3 and H^3 . This result was proved and discussed in detail in [20, 38], and this is why in this paper we will limit to a brief account only.

Theorem 2.1. *There is a bijection between flat connections in $\mathcal{M}_{\text{flat}}(S_a, \text{PSU}(2))$ and the convex constant curvature tetrahedron geometries in $3d$, as far as the non-degenerate geometry is concerned.⁸*

The correspondence applies to both spherical and hyperbolic tetrahedra. Both positive and negative constant curvature geometries are included in $\mathcal{M}_{\text{flat}}(S_a, \text{PSU}(2))$.

The theorem repose on two main observations: (i) the fundamental group of the four-holed three-sphere is isomorphic to that of a tetrahedron’s one-skeleton, and both are defined by a closure constraint; and (ii) in the flat case, a tetrahedron’s geometry can be fully reconstructed from four vectors which add up to zero, once these are interpreted as the tetrahedron’s face normals of lengths equal to the respective face areas. The previous two points are related by the observation that the spin-connection holonomy around the perimeter of a surface flatly-embedded in a homogeneous space contains informations about both the area and the orientation of the surface, and therefore the curved-space closure constraint could be a sound generalization of (ii).

In fact, observation (ii) is a special, almost trivial, case of a more general classic result by H. Minkowski [?], known as Minkowski’s reconstruction theorem. This states that an N -tuple of vectors that sum to zero corresponds to the set of face vectors of one, and only one, convex polyhedron with N faces. The convexity hypothesis, redundant in the case of flat tetrahedra, turns out to be crucial in the curved-space generalization of Minkowski’s reconstruction.

⁸The non-degenerate tetrahedron geometry is dense in $\mathcal{M}_{\text{flat}}(S_a, \text{PSU}(2))$.

Before going into any further detail, let us define what we mean by a simplicial geometry flatly embedded in a unit three-sphere S^3 . The zero-simplices (vertices) are 4 points on S^3 . The one-simplices are the *shortest* geodetic arcs connecting 2 zero-simplices. These are given by arcs of the unit circle, which are great circles of S^3 . Notice that the restriction on the length of the geodetic arc is related to the will of considering only convex simplices, which will turn out to be crucial for the uniqueness part of the reconstruction theorem. Triplets of points identify uniquely a great 2-sphere in S^3 . As portions of great circles are geodetic arcs, portions of a great two-sphere are flatly embedded in S^3 (see below). In particular this means that vectors orthogonal to the surface stay such. It is not hard to show that the intersection of two flatly embedded surfaces is a geodetic arc. It is hence meaningful to talk about the curved triangular face of the tetrahedron (two-simplices) as the convex spherical triangle having as vertices three zero-simplices. Finally, the tetrahedron itself is the convex hull defined by its four faces.

The simplest way to visualize this construction is to consider the unit three-sphere as embedded in one more dimension. Then, the edges of the tetrahedron are defined by intersection of the three-sphere with the unique plane passing through the origin and two of the tetrahedron's vertices. Similarly, the tetrahedron's faces are given by the intersection of the three-sphere and the unique hyperplane passing through the origin and three of the tetrahedron's vertices.

This construction makes it obvious how to generalize the definitions to the hyperbolic, higher dimensional, and Lorentzian cases.

The connection between these geometries and spin-connection holonomies is made explicit by the following

Lemma 2.2. *Let $(\mathfrak{M}_4, g_{\alpha\beta})$ be a 4-dimensional Lorentzian spacetime of constant curvature Λ , and e_α^I an orthonormal frame on \mathfrak{M}_4 . That is $g_{\alpha\beta} = \eta_{IJ} e_\alpha^I e_\beta^J$, where $\eta_{IJ} = \text{diag}(-1, 1, 1, 1)$. Consider a 2-dimensional space-like surface \mathfrak{f} with boundary flatly embedded in \mathfrak{M}_4 , i.e. such that its unit time-like normal vectors u^α and unit space-like normal vector n^α satisfy*

$$h_\alpha^{\alpha'} h_\beta^{\beta'} \nabla_{\alpha'} u_{\beta'} = 0 = h_\alpha^{\alpha'} h_\beta^{\beta'} \nabla_{\alpha'} n_{\beta'} \quad (2.17)$$

where $h_\alpha^{\alpha'}$ is the projector onto the 2-surface and ∇_α is the torsion-free derivative satisfying $\nabla_\gamma g_{\alpha\beta} = 0$ (Levi-Civita connection).

Then, the holonomy of spin connection ω_{Spin} along ∂f based at $O \in \partial \mathfrak{f}$ is a function of the bivector $\varepsilon^{\alpha\beta} e_\alpha^I e_\beta^J$ at O and of the surface area a :⁹

$$U_{\partial f}(\omega_{\text{Spin}}) = \exp \left\{ \frac{\Lambda}{3} a \left[\varepsilon^{\alpha\beta} e_\alpha e_\beta \right]_+ (O) \right\} \in \text{SL}(2, \mathbb{C}), \quad (2.18)$$

where $\varepsilon_{\alpha\beta}$ is the area element of f and $\left[\varepsilon^{\alpha\beta} e_\alpha e_\beta \right]_+ \in \mathfrak{sl}_2 \mathbb{C}$ is the self-dual part of the bivector.

After a partial gauge fixing at the base point O , such that $e_0^\alpha = \delta_0^\alpha$ is the unit time-like normal of \mathfrak{f} (time gauge), then

$$U_{\partial f}^0(\omega_{\text{Spin}}) = \exp \left[\frac{\Lambda}{3} a \hat{n} \cdot \vec{\tau} \right] \in \text{SU}(2). \quad (2.19)$$

where \hat{n} is a unit 3-vector in \mathbb{R}^3 , such that $\sum_{i=1}^3 \hat{n}^i e_i^\alpha (O)$ is the space-like normal to \mathfrak{f} .

The proof of the lemma is a straightforward calculation in differential geometry. It is given for completeness in Appendix A (see also [20]).

The lemma is completely general and holds for any flatly embedded surface \mathfrak{f} . In particular it must hold for each face of the tetrahedron. Label the vertices of the four-simplex flatly embedded in (A)dS by

⁹Here both the dimensionful quantities Λ and a are defined with respect to a certain length unit. Only the combination Λa is dimensionless and unit independent.

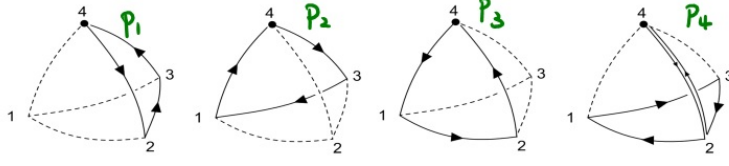


Figure 3. Paths on a tetrahedron. The edge (2, 4) is call a “special edge”.

$a, b, \dots \in \{1, \dots, 5\}$, and the tetrahedron opposite to vertex a by the same label. The triangle shared by tetrahedra a and b is thus labeled by (ab) . Name its boundary Δ_{ab} . With this notation,

$$U_{ab} \equiv U_{\partial\Delta_{ab}}^0(\omega_{\text{Spin}}) = \exp \left[\frac{\Lambda}{3} a_{ab} \hat{n}_{ab} \cdot \vec{\tau} \right] \in \text{SU}(2), \quad (2.20)$$

where a_{ab} is the area of the triangle (ab) , \hat{n}_{ab} its spacelike normal (expressed within the frame e_α^I at O), and $\Lambda \geq 0$ is the cosmological constant associated with (A)dS.

The idea is now to find a relation between the U_{ab} ’s and the $H_b(a)$ ’s of equation 2.12. In order to do this we need to make the isomorphism between the fundamental groups of the 4-holed sphere \mathcal{S}_a and that of the a -th tetrahedron’s 1-skeleton τ_a . It is important to notice that there is no canonical isomorphism. We will come back to this point when dealing with the reconstruction of the 4-simplex geometry. For the moment, we limit our study to a single tetrahedron, say $a = 5$ and hence drop the relative label. In this way, triangles are label simply by their opposite vertex within this tetrahedron.

Name the isomorphism between the two fundamental groups of \mathcal{S}_5 and τ_5 , \mathcal{I}_5 :

$$\mathcal{I}_5 : \pi_1(\tau_5) \longrightarrow \pi_1(\mathcal{S}_5). \quad (2.21)$$

To specify \mathcal{I}_5 , consider the basis of $\pi_1(\tau_5)$ formed by the set of four paths $\{p_a\}$ depicted in figure 3. Each p_a goes around a face of the tetrahedron only once. Hence, we require these paths to be in a 1-to-1 correspondence with the l_a previously introduced, which in turn go around each puncture of \mathcal{S}_5 only once:

$$\mathcal{I}_5(p_a) = l_a. \quad (2.22)$$

This is possible thanks to the fact that both sets of paths satisfy the defining constraints

$$p_4 p_3 p_2 p_1 = e \quad \text{and} \quad l_4 l_3 l_2 l_1 = e \quad (2.23)$$

Following [20, 38], we name the paths $\{p_a\}$, ‘simple paths’. The name comes from the fact that this is arguably the shortest set of paths satisfying the defining constraint (up to relabeling of the vertices). It is also clear that edge (42) is singled out by this choice of paths (see [20] for the explicit role it plays in the reconstruction theorem). We name it the ‘special edge’.

Notice that the simple paths are such that the faces are circled around counterclockwise (as seen from the outside of the tetrahedron). Adopting a right-handed convention, this means that the normals to the triangles have to be understood as outgoing.

This isomorphism allows us to interpret the holonomies of a flat connection ω_{flat} on \mathcal{S}_5 as the parallel transports of a spin-connection ω_{Spin} on τ_5 :

$$\begin{array}{ccc} \pi_1(\mathcal{S}_5) & \xleftarrow{\mathcal{I}_5} & \pi_1(\tau_5) \\ \omega_{\text{flat}} \searrow & & \swarrow \omega_{\text{Spin}} \end{array}$$

$$\left\langle U_1, \dots, U_4 \in \text{SU}(2)^{\otimes 4} \mid U_4 \cdots U_1 = 1 \right\rangle / \text{conjugation} \quad (2.24)$$

at least provided we find a canonical lift of $H_a \in \text{PSU}(2)$ to $\text{SU}(2)$. The prescription for the canonical lift is actually provided by the convexity condition. Let us explain how this works.

An element $H \in \text{PSU}(2)$ is given by the equivalence class formed by the following two elements of $\text{SU}(2)$:

$$\exp[a \hat{n} \cdot \vec{\tau}] \sim -\exp[a \hat{n} \cdot \vec{\tau}] = \exp[(2\pi - a)(-\hat{n}) \cdot \vec{\tau}] \quad (2.25)$$

for some $a \in [0, 2\pi]$ and $\hat{n} \in \mathbb{S}^2$. The previous correspondence suggests to interpret

$$\hat{n}, \text{ or } -\hat{n}, \text{ as } \text{sgn}(\Lambda)\hat{n} \quad \text{and} \quad a, \text{ or } (2\pi - a) \text{ respectively, as } \frac{|\Lambda|}{3}a. \quad (2.26)$$

Using the simple path convention of outgoing normals, and the tetrahedron's convexity, one sees that the triple products $\hat{n}_a \cdot (\hat{n}_b \times \hat{n}_c)$, when the labels $\{a, b, c\}$ are properly ordered, must all be positive¹⁰ (e.g. at vertex 4, $\hat{n}_1 \cdot (\hat{n}_2 \times \hat{n}_3) > 0$). It is hence clear that the convexity conditions fully determine the lift from $\text{PSU}(2)$ to $\text{SU}(2)$, at least up to a global sign, equal to $\text{sgn}(\Lambda)$. In turn, this sign can be determined from the \hat{n}_a so calculated.

This goes as follows. From the \hat{n}_a calculate the scalar products $\cos \theta_{ab} \equiv \hat{n}_a \cdot \hat{n}_b$. Notice that these quantities are insensitive to the global sign ambiguity associated with $\text{sgn}(\Lambda)$ itself. These scalar products¹¹ are nothing but the (external) dihedral angles of the tetrahedron. It is a classical result in discrete geometry, that the Gram matrix

$$(\text{Gram})_{ab} = -\cos \theta_{ab} \quad (2.27)$$

contains all the information needed to reconstruct the tetrahedron's geometry. In particular

$$\text{sgn}(\det(\text{Gram})) = \text{sgn}(\Lambda). \quad (2.28)$$

At this stage, to conclude the proof of the reconstruction theorem, one only needs to prove the consistency of the geometry reconstructed from the Gram matrix and the areas implicitly contained in the original group elements. This can be done for example via a counting argument. Again, for all the details of the proof see [20].

For future reference, we note here the formula interpreting the transverse holonomies H_{ab} as the spin-connection holonomies around the face (ab) of the four-simplex:

$$H_b(a) = \exp \left[\frac{\Lambda}{4} a_{ab} \hat{n}_{ab} \cdot \vec{\tau} \right]. \quad (2.29)$$

2.4 Flat Connections on 3-Manifold and Curved 4-Simplex Geometries

In the same way as the discussion for the commuting diagram (2.24), we consider the fundamental group for the 1-skeleton of an abstract 4-simplex, see figure 4, which we denote by $\pi_1(\sigma_4)$, σ_4 being the 4-simplex's 1-skeleton. The generator are the close paths p_{ab} along the 1-skeleton and circling around a single triangle Δ_{ab} . A convenient set of paths p_{ab} or p_{ab}^{-1} are specified by the sets of simple paths for all 5 tetrahedra, which are counter-clockwise viewed from outside. All the closed paths p_{ab} are based at the same point, which we choose to be vertex 1 of the 4-simplex.

Explicitly, we choose the paths as follows:

¹⁰Notice that there might be the need of a parallel transport. This happens when one has to compare the normal relative to face 4 to the others. This is the parallel transport of a 3-vector, it makes use of the vector representation of the H_a 's, and is hence immune to the ambiguity we are trying to solve here. See ?? for details.

¹¹Again, in some cases a parallel transport of the normals is needed before taking the scalar product. In this case, using the flat-embedding condition, it is not hard to convince oneself that the only dihedral angle needing a 'twisted' formula is $\cos \theta_{24} = \hat{n}_4 \cdot (\mathbf{H}_3 \hat{n}_2)$. Here, $\mathbf{H}_3 \in \mathbf{SO}(3)$ is the vector representation of H_3 . See [20].

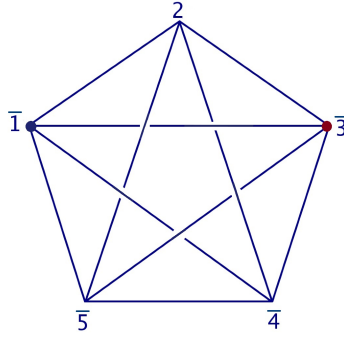


Figure 4. An abstract 4-simplex, whose vertices are labeled by $\bar{1}, \dots, \bar{5}$. The tetrahedron tetra_a denotes the tetrahedron which doesn't have the vertex \bar{a} . The triangle Δ_{ab} (resp. Δ_{ba}) denotes the triangle belonging to tetra_a (resp. tetra_b) shared by tetra_a and tetra_b . The edges are denoted by (\bar{a}, \bar{b}) oriented from \bar{b} to \bar{a} .

- τ_2 has as special edge (31), and its closure relation is¹²

$$p_{21}^{-1} p_{24} p_{23} p_{25} = e. \quad (2.30)$$

- τ_3 has as special edge (51), and its closure relation is

$$p_{32}^{-1} p_{31}^{-1} p_{34} p_{35} = e. \quad (2.31)$$

- τ_4 has as special edge (31), and its closure relation is

$$p_{43}^{-1} p_{42}^{-1} p_{41}^{-1} p_{45} = e. \quad (2.32)$$

- τ_5 has as special edge (31), and its closure relation is

$$p_{52}^{-1} p_{53}^{-1} p_{54}^{-1} p_{51}^{-1} = e. \quad (2.33)$$

- τ_1 is the 'special tetrahedron' which is non-adjacent to the base vertex 1. All the paths associated to τ_1 travel from 1 to 3 along (31), then circle around the relevant triangle of τ_1 as in figure 3, and finally go back from 3 to 1 along (13). When we draw the paths on τ_1 starting and ending at 3, the special edge is (5, 3). The closure relation is then

$$p_{14} p_{13} p_{12} p_{15} = e. \quad (2.34)$$

The above list specifies all the (closed) paths p_{ab} . One can check the following properties: (i) $p_{ab} = p_{ba}$ for $(a, b) \neq (1, 3)$, while (ii) the relation between p_{13} and p_{31} is $[53][31]p_{13}[13][35] = [51]p_{31}[15]$, where $[ab]$ indicates the path along the edge ab . Equivalently,

$$p_{13} = p_{24} p_{31} p_{24}^{-1} \quad (2.35)$$

where $p_{24} = [13][35][51]$.

The fundamental group $\pi_1(\sigma_4)$ is generated by the closed paths p_{ab} subjected to the set of closure relations Eqs.(2.30) - (2.34) together with the relation (2.35). A quick comparison shows that $\pi_1(\sigma_4)$ is isomorphic to $\pi_1(S^3 \setminus \Gamma_5)$. In fact, the relations above for the p_{ab} 's generating $\pi_1(\sigma_4)$ are identical to the

¹²Note that all the paths $p_{21}^{-1}, p_{24}, p_{23}, p_{25}$ are closed paths circling around a single triangle in a counter-clockwise fashion when viewed from the outside of the tetrahedron. The same holds for Eqs.(2.31) - (2.34).

relations associated to the generators l_{ab} of $\pi_1(S^3 \setminus \Gamma_5)$ (see Section 2.1). The isomorphism maps the generators of $\pi_1(S^3 \setminus \Gamma_5)$ to the generators of $\pi_1(\text{simplex})$, which delivers the flat connection on $S^3 \setminus \Gamma_5$ to the spin connection as a representation of $\pi_1(\text{simplex})$.

If we require that the isomorphism $\mathcal{I} : \pi_1(\sigma_4) \rightarrow \pi_1(S^3 \setminus \Gamma_5)$ maps the counterclockwise simple paths (as p_{ab}) to the loop generators in $\pi_1(S^3 \setminus \Gamma_5)$ oriented in a right-handed manner (as l_{ab}) according to the orientation of the edges $\ell_{ab} \subset \Gamma_5$, then the isomorphism \mathcal{I} is unique in the following sense:

Lemma 2.3. *A map $\iota : a \mapsto \tau_a$ identifying a vertex in Γ_5 with a tetrahedron on the boundary of the 4-simplex, induces an identification between the edges ℓ_{ab} and the triangles $\Delta_{ab} = \tau_a \cap \tau_b$. Given an isomorphism $\mathcal{I} : \pi_1(\sigma_4) \rightarrow \pi_1(S^3 \setminus \Gamma_5)$ such that $\mathcal{I}(p_{ab}) = l'_{ab}$ is a loop generator in $\pi_1(S^3 \setminus \Gamma_5)$ transverse to the edge ℓ_{ab} close to the vertex a , requiring that l'_{ab} cycles ℓ_{ab} in a right-handed manner according to the orientation of ℓ_{ab} ¹³, the isomorphism \mathcal{I} is unique with $\mathcal{I}(p_{ab}) = l_{ab}$ being the generator for the presentation in Section 2.1 associated to the projection FIG.2 of Γ_5 on a plane.*

Proof: The set of loops $\mathcal{I}(p_{ab}) = l'_{ab}$, whose common base point could be anywhere in $S^3 \setminus \Gamma_5$, can be understood as the generators of a generalized Wirtinger presentation of $\pi_1(S^3 \setminus \Gamma_5)$ from a certain projection of Γ_5 on a plane, which could be different from FIG.2. However $p_{ab} = p_{ba}$ implies $l'_{ab} = l'_{ba}$ for $(a, b) \neq (1, 3)$ by the isomorphism \mathcal{I} . It means that in this projection of Γ_5 , the loops l'_{ab} for $(a, b) \neq (1, 3)$ can continuously be deformed through the whole edge ℓ_{ab} without meeting a crossing. Therefore the crossing only happens between ℓ_{13} and ℓ_{24} . Then the projection is either (a) as in FIG.2, with ℓ_{24} over-crossing ℓ_{13} , or (b) as it would appear if FIG.2 was viewed from the back, i.e. with ℓ_{24} under-crossing ℓ_{13} . Without loss of generality, we assume the base point of l'_{ab} is in front of the projected graph in both cases (a) and (b). Furthermore the relations Eqs.(2.30) - (2.34) imply the same relations for l'_{ab} up to cyclic permutation. These relations for l'_{ab} imply that in the case (a), each loop l'_{ab} circles ℓ_{ab} in a right-handed manner (as in Eq.(2.3)) with respect to the orientation of ℓ_{ab} , while in case (b) each loop l'_{ab} circles ℓ_{ab} in a left-handed way. Both (a) and (b) imply $l'_{13} = l'_{24} l'_{31} l'_{24}^{-1}$. However, (b) is ruled out by the requirement that l'_{ab} cycles ℓ_{ab} in a right-handed manner. Therefore we conclude that the case (a) is singled out, and $l'_{ab} = l_{ab}$. \square

The identification map $\iota : a \mapsto \tau_a$ produces the numbering of tetrahedra (or vertices) of an abstract 4-simplex from the numbering of Γ_5 vertices, by the convention that τ_a labels the tetrahedron which does *not* contain the vertex \bar{a} in FIG.4. Given such an identification, we have the following diagram if the 4-simplex is embedded in a geometrical space with spin connection ω_{spin} :

$$\begin{array}{ccc} \pi_1(S^3 \setminus \Gamma_5) & \xleftarrow{\mathcal{I}} & \pi_1(\sigma_4) \\ \omega_{\text{flat}} \searrow & & \swarrow \omega_{\text{spin}} \\ \langle \{\tilde{H}_{ab}\} \mid \text{Eqs. (2.12a) - (2.6)} \rangle / \text{conjugation} & & \end{array} \quad (2.36)$$

where the isomorphism \mathcal{I} is unique in the sense of the previous Lemma. The isomorphism \mathcal{I} determines by restriction the isomorphisms \mathcal{I}_a associated to each of the five tetrahedra. The isomorphisms \mathcal{I}_a in the diagram (2.24) are unique if embedded in a 4-simplex context.

The representation ω_{spin} associates to the set of paths p_{ab} the holonomies of an $\text{SL}(2, \mathbb{C})$ spin connection:

$$\omega_{\text{spin}}(p_{ab}) = U_{ab}. \quad (2.37)$$

On the other hand, the flat connection representation on $S^3 \setminus \Gamma_5$ discussed in Section 2.1, gives

$$\omega_{\text{flat}}(l_{ab}) = \tilde{H}_{ab}. \quad (2.38)$$

¹³The orientation condition for l'_{ab} is corresponding to the counter-clockwise-ness of the paths p_{ab} or p_{ab}^{-1} in Eqs.(2.30) - (2.34).

The above diagram gives us

$$\pm \omega_{\text{spin}} = \omega_{\text{flat}} \circ \mathcal{I}. \quad (2.39)$$

and hence

$$\pm U_{ab} = \tilde{H}_{ab} \quad (2.40)$$

This relation allows us to interpret the holonomies of a flat connection \tilde{H}_{ab} to be the holonomies of a spin connection along the paths \mathfrak{p}_{ab} around the 1-skeleton of an embedded 4-simplex. The \pm sign comes from the fact that Theorem 2.1 holds for PSU(2) flat connection, and H_{ab} is identified with the spin connection U_{ab} up to a sign, as discussed in Section 2.3.

Here we are relating the flat connection A on $S^3 \setminus \Gamma_5$ to the geometry of a 4-simplex embedded in a constant curvature (Lorentzian) spacetime, whose boundary tetrahedra are constant curvature spacelike tetrahedra with flatly embedded surfaces. The flat connection A on $S^3 \setminus \Gamma_5$ is taken to satisfy the boundary condition of Section ??, which give us $H_b(a) = g_a^{-1} \tilde{H}_{ab} g_a \in \text{SU}(2)$. In turn, the reconstruction theorem of Section 2.3 grants us that and that the equation $\prod_b H_b(a) = 1$ associates to τ_a the geometry of a non-degenerate convex spacelike tetrahedron with constant curvature Λ_a ¹⁴, and hence the interpretation of the $H_b(a)$ in terms of face vectors $a_{ab} \hat{n}_{ab}$:

$$H_b(a) = \exp \left[\frac{\Lambda_a}{3} a_{ab} \hat{n}_{ab} \cdot \vec{\tau} \right]. \quad (2.41)$$

where $\Lambda_a = \pm_a |\Lambda|$. For future convenience we introduce

$$\nu_a = \text{sgn } \Lambda_a \quad \text{and} \quad \nu = \text{sgn } \Lambda. \quad (2.42)$$

Λ is a constant for all τ_a , whose sign will be determined shortly. This constant introduces a length unit. a_{ab} are the areas of the convex constant curvature tetrahedron. Note that at this stage we do not know whether the boundary data induce a sign ν_a constant throughout the 4-simplex. However, this is something we will prove to follow from the requirement that the boundary data are given by the boundary value of A .

When the time-like normal of tetrahedron a is gauge fixed to be $(1, 0, 0, 0)^T$, $\hat{n}_{ab} = \nu_a \hat{n}_{ab}$ ¹⁵ is the spatial normal 3-vector to the triangle Δ_{ab} , parallel transported at the base point of \mathfrak{p}_{ab} , i.e. at vertex 1 of the 4-simplex. In fact, a parallel transportation (depending on the pattern of \mathfrak{p}_{ab}) is needed when Δ_{ab} is not adjacent to 1.

Until now we have studied only the geometry of the single tetrahedra. It is now time to focus on the geometry of the full 4-simplex, and to show how this can be reconstructed from the knowledge of the holonomies $H_b(a)$ and G_{ab} alone.

The group elements $\pm g_a \in \text{SL}(2, \mathbb{C})$, which allow to put into the time-gauge the five tetrahedra, give the Lorentz frame of the four surfaces contained in each tetrahedron. By Lemma 2.2, $\tilde{H}_{ab} = g_a H_b(a) g_a^{-1}$ can be interpreted as

$$\tilde{H}_{ab} = \exp \left[\frac{\Lambda_a}{3} a_{ab} \mathcal{E}_{ab}(1) \right] \quad (2.43)$$

where $\mathcal{E}_{ab}(\bar{1})$ is the surface area bivector located at $\bar{1}$:

$$\mathcal{E}_{ab} = \left[e^{\alpha\beta} e_\alpha e_\beta \right]_+ (1) \text{ of } \Delta_{ab}. \quad (2.44)$$

Note that $\mathcal{E}_{ab}(1)$ is related to $\hat{n}_{ab} \cdot \vec{\tau}$ by

$$\mathcal{E}_{ab}(1) = -g_a (\hat{n}_{ab} \cdot \vec{\tau}) g_a^{-1}. \quad (2.45)$$

¹⁴We only consider the boundary data corresponds to nondegenerate tetrahedron geometry, which is dense in the space of boundary data.

¹⁵Here $\mathcal{E}_{ab} \hat{n}_{ab}$ is the outward-pointing normal of tetra_a.

The set of $\mathcal{E}_{ab}(1)$ is defined up to a simultaneous adjoint action of $\text{SL}(2, \mathbb{C})$. $\mathcal{E}_{ab}(1)$ is the surface area bivector located at 1, and a simultaneous adjoint action of $\text{SL}(2, \mathbb{C})$ is nothing but a local Lorentz transformation in the base frame at 1.

Similar to what happens for the \hat{n}_{ab} , a parallel transportation (dependent on the specific pattern of the \mathfrak{p}_{ab}) relates $\mathcal{E}_{ab}(1)$ to the actual bivector on Δ_{ab} , whenever Δ_{ab} is not adjacent to 1. For the Δ_{ab} 's adjacent to 3, their bivectors are given by $\mathcal{E}_{ab}(3) = \Omega[31]\mathcal{E}_{ab}(1)\Omega[31]^{-1}$ where $\Omega[\vec{a}, \vec{b}] \in \text{SL}(2, \mathbb{C})$ is the holonomy of spin connection ω_{Spin} along the edge (ab) .

Note that the tetrahedra reconstructions do not grant automatically that the areas of the triangles Δ_{ab} as seen from tetrahedra a and b coincide. The reason is the \mathfrak{a}_{ab} versus $2\pi - \mathfrak{a}_{ab}$ ambiguity. However, the following theorem guarantees that this is not the case:

Theorem 2.4. *A dense subset of flat connection $A \in \mathcal{M}_{\text{flat}}^{\text{BC}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C}))$, i.e. such that their restriction to the boundary $\mathfrak{A} \in \mathcal{M}_{\text{flat}}(\Sigma_6, \text{SL}(2, \mathbb{C}))$ satisfies the boundary conditions corresponding to 5 non-degenerate convex constant curvature tetrahedra, determine each a unique non-degenerate convex Lorentzian 4-simplex geometry with constant curvature Λ , whose boundary geometry is consistent with the tetrahedron geometry determined by \mathfrak{A} .*

The proof of the theorem (see [38]) is analogous to that of the three-dimensional case, and also employs the reconstruction of the 4-simplex's Gram matrix

$$\text{Gram}_4 \equiv \cosh \Theta_{ab}, \quad (2.46)$$

where Θ_{ab} are the boost dihedral angles of the four-simplex. This contains all the information needed to reconstruct the 4-simplex geometry, here included the sign of its curvature. It is calculated via the equation

$$\cosh \Theta_{ab} = -u_I(\hat{G}_{ab})^I{}_J u^J, \quad (2.47)$$

where $u^I = (1, 0, 0, 0)^T$, and $\hat{G}_{ab} \in \text{SO}^+(1, 3)$ is the vectorial representation of $G_{ab} \in \text{SL}(2, \mathbb{C})$.

The non-degeneracy condition corresponds to the requirement that the connection does *not* produce G_{ab} such that $u_I(\hat{G}_{ab})^I{}_J u^J = 1$.

Notice that the theorem implies in particular that all the five boundary tetrahedra share the same sign of the curvature, hence

$$\nu_a = \nu = \text{sgn } \Lambda \quad (2.48)$$

is a *global* sign.

The theorem also allows to reconstruct the meaning of the rotation part of G_{ab} . This is associated to the plane of the triangle Δ_{ab} , and corresponds to the relative rotation by an angle θ_{ab} between the frames of Δ_{ab} as seen from tetrahedra a and b .

2.5 Parity Pairs

The boundary condition requires that the boundary value of the flat connection in $\mathcal{M}_{\text{flat}}^{\text{BC}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C}))$, which is a flat connection in $\mathcal{M}_{\text{flat}}^{\text{BC}}(\Sigma_6, \text{SL}(2, \mathbb{C}))$, reduces to an $\text{SU}(2)$ flat connection on 4-holed sphere. The boundary data on Σ_6 are completely determined by specifying at the same time:

- i) the conjugacy classes of the holonomies around a path ℓ_{ab} transverse to each edge of Γ_5 . This is equivalent to specifying (one of) the eigenvalues x_{ab} of such holonomies. In particular, the boundary conditions impose that $x_{ab} \in \text{U}(1)$, instead of being a general complex number;
- ii) the eigenvalues x_a and x_a'' of the products of two pairs of holonomies $(c_{ab}, c_{ab'})$ and $(c_{ab}, c_{ab''})$, encircling three different edges adjacent to the same vertex a . Name the paths associated to the above compositions c_a and c_a'' . Again, x_a and x_a'' must be complex numbers of unit norm, i.e. $x_a, x_a'' \in \text{U}(1)$.

In the next section, we will discuss why it is far more convenient to substitute x_a'' with a coordinate y_a , which turns out to be canonically conjugated (in the sense of symplectic geometry) to x_a . In terms of these variables, known as Fenchel-Nielsen length and twist respectively, the boundary conditions read once again $x_a, y_a \in U(1)$.

The boundary data $\{x_{ab}; x_a, y_a\} \subset U(1)$ fully specify the $SU(2)$ flat connections on the five 4-punctured spheres $\{\mathcal{S}_a\}_{a=1,\dots,5}$. The geometrical reconstruction theorems discussed above imply that these same data encode completely the geometry of five geometrical constant-curvature tetrahedra. These tetrahedra are characterized by the fact that the value of their faces' areas are shared by couples of tetrahedra. This is because, geometrically, the $\{x_{ab}\}_b$ encode the areas of the faces of tetrahedron a . On the other hand the $\{x_a, y_a\}$ fix the remaining two degrees of freedom (a tetrahedron is determined by 6 independent edge lengths). At this level nothing is enforcing the fact that the shapes of the faces of equal areas in two different tetrahedra are the same.

Note that at fixed areas, the space of tetrahedra parametrized by (x_a, y_a) turns out to carry a natural symplectic structure [20], such that the logarithms of these variables are conjugated. We will come back on this fact in the next section.

Denote a given value of the boundary data $\{x_{ab}; x_a, y_a\}$ by $\{\hat{x}_{ab}; \hat{x}_a, \hat{y}_a\}$.

The following questions and their answers turn out to be interesting and useful in the later analysis:

Q1 Does it always exist a flat connection $A \in \mathcal{M}_{\text{flat}}^{\text{BC}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C}))$ whose boundary value is consistent with a given set of the boundary data $\{\hat{x}_{ab}; \hat{x}_a, \hat{y}_a\}$?

Q2 Provided such a consistent flat connection exists, is it uniquely determined by the $\{\hat{x}_{ab}; \hat{x}_a, \hat{y}_a\}$?

Both of the above questions have the negative answers. Let us explain why.

A1 A generic flat connection in $\mathcal{M}_{\text{flat}}^{\text{BC}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C}))$ satisfies the hypothesis of Theorem 2.4 and hence corresponds to a geometric four-simplex. However, as we discussed above, within the boundary data $\{\hat{x}_{ab}; \hat{x}_a, \hat{y}_a\}$ there is nothing that guarantees the correspondence of the shapes of the triangular faces. Hence, not every set of boundary conditions $\{\hat{x}_{ab}; \hat{x}_a, \hat{y}_a\}$ is the boundary of a flat connection in $\mathcal{M}_{\text{flat}}^{\text{BC}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C}))$.

A2 Consider a set of boundary data $\{\hat{x}_{ab}; \hat{x}_a, \hat{y}_a\}$, and a flat connection $A \in \mathcal{M}_{\text{flat}}^{\text{BC}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C}))$ consistent with them. Theorem 2.4 states that A corresponds uniquely to a geometric 4-simplex σ_4 . However, as the next theorem shows, it is easy to produce from A another flat connection $\tilde{A} \in \mathcal{M}_{\text{flat}}^{\text{BC}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C}))$ whose boundary value is consistent with $\{\hat{x}_{ab}; \hat{x}_a, \hat{y}_a\}$. Notice that this does not mean that A and \tilde{A} must have the same boundary values when restricted to Σ_6 , since the data $\{\hat{x}_{ab}; \hat{x}_a, \hat{y}_a\}$ do not contain informations about the *longitudinal* holonomies G_{ab} . In fact, it turns out that A and \tilde{A} correspond to different constant curvature 4-simplex σ_4 and $\tilde{\sigma}_4$ related by a parity inversion, and $\tilde{G}_{ab} = G_{ab}^{-1\dagger}$. In analogy with the previous discussion we can introduce the variables y_{ab} , conjugated to the x_{ab} , in order to have a complete set of coordinates on $\mathcal{M}_{\text{flat}}(\Sigma_6, \text{SL}(2, \mathbb{C}))$. Hence the parity pair has coordinates $\{\hat{x}_{ab}, \hat{y}_{ab}; \hat{x}_a, \hat{y}_a\}$ and $\{\hat{x}_{ab}, \overline{\hat{y}_{ab}}; \hat{x}_a, \hat{y}_a\}$, where $\overline{\hat{y}_{ab}} = 1/\hat{y}_{ab}$, the bar standing for complex conjugation.

Theorem 2.5. *Given a set of boundary data $[\hat{x}_{ab}; \hat{x}_a, \hat{y}_a]$ corresponding geometrically to 5 constant curvature tetrahedra forming the boundary of a constant curvature 4-simplex, there exists exactly 2 flat connections $A, \tilde{A} \in \mathcal{M}_{\text{flat}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C}))$ on the graph complement 3-manifold, whose boundary values are consistent with $[\hat{x}_{ab}; \hat{x}_a, \hat{y}_a]$. A and \tilde{A} correspond to the constant curvature 4-simplices σ and $\tilde{\sigma}$ which have the same geometry but have different 4d orientation. The pair of A, \tilde{A} is called a “parity pair”.*

The proof can be found in [20]. The existence of the parity pair A, \tilde{A} is natural, because A, \tilde{A} are complex conjugate to each other with respect to the complex structure on $\mathcal{M}_{\text{flat}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C}))$ induced from the complex group $\text{SL}(2, \mathbb{C})$ ¹⁶. So the boundary values of A, \tilde{A} give the same $\text{SU}(2)$ flat connection on each 4-holed sphere S_a , which implies that they give the same data $[\hat{x}_{ab}; \hat{x}_a, \hat{y}_a]$.

3 Complex Fenchel-Nielsen (FN) Coordinate and Simplicial Geometry

Theorem 2.4 shown that an $\text{SL}(2, \mathbb{C})$ flat connection on $S^3 \setminus \Gamma_5$, satisfying the boundary condition, determines uniquely a geometrical 4-simplex with constant curvature. Such a correspondence leads to the geometrical interpretation of the complex FN coordinate, for each flat connection in $\mathcal{M}_{\text{flat}}(\Sigma_6, \text{SL}(2, \mathbb{C}))$ which can be extended to 3-dimensions as a flat connection in $\mathcal{M}_{\text{flat}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C}))$. To find explicitly the geometrical interpretation of the complex FN coordinate, we need its relation with the holonomies (and their eigenvectors) of flat connection on Σ_6 , which has been studied in [32, 36]. Here we re-drive the relation with respect to the meridian and longitude holonomies H_{ab}, G_{ab} , which is heavily used in the geometrical reconstruction. It is immediate that the FN length variable x_{ab} , being the eigenvalue of the meridian H_{ab} , relates the triangle areas of the 4-simplex in the geometrical interpretation. Then the task in the following is to understand the relation between the complex twist variable y_{ab} and 4-simplex geometry.

Let's consider arbitrary 2 vertices connected by an edge in Γ_5 . It corresponds to gluing two 4-holed spheres through a pair of holes on the boundary Σ_6 of the graph complement (see Fig. 5). Given an flat connection $\mathfrak{A} \in \mathcal{M}_{\text{flat}}(\Sigma_6, \text{SL}(2, \mathbb{C}))$ satisfying the boundary condition on Σ_6 , we choose a set of loops based at one point p_a or p_b on each 4-holed sphere S_a or S_b , such that the holonomies of \mathfrak{A} along the loops gives the $\text{SU}(2)$ representations of $\pi_1(S_a)$ and $\pi_1(S_b)$ respectively. The holonomy along the loop transverse to the edge ℓ_{ab} is H_{ab} (resp. H_{ba}) if it is on S_a (resp. S_b). We draw a longitude curve connecting the base points p_a and p_b , such that its tangent vectors at the two base points are linear independent to the tangent vectors of the transverse loops for H_{ab} and H_{ba} . The holonomy of \mathfrak{A} along the longitude curve is G_{ab} (defined in Eq.(??)). Note that given the above set of curves, the gauge transformations at the base points p_a and p_b reduce to $\text{SU}(2)$ gauge transformations, in order to preserve the $\text{SU}(2)$ representations of $\pi_1(S_a)$ and $\pi_1(S_b)$.

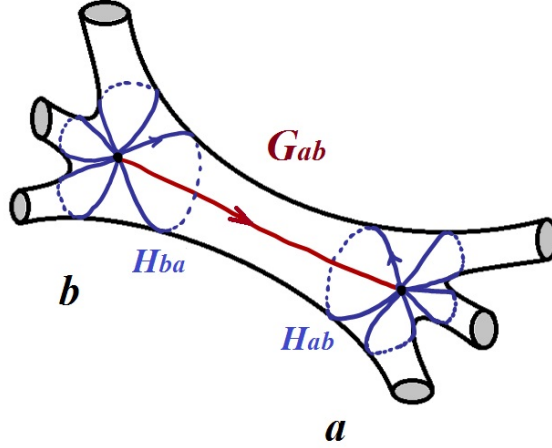


Figure 5. A part of Σ_6 which contains two 4-holed spheres S_a and S_b , corresponding to 2 vertices a and b connected by an edge ℓ_{ab} in Γ_5 . The holonomies H_{ab}, H_{ba} , etc, along the blue curves give the $\text{SU}(2)$ representations of $\pi_1(S_a)$ and $\pi_1(S_b)$. The longitude holonomy along the red curve connecting the two base points is denoted by G_{ab} .

¹⁶Namely $A = A^j \tau_j$ and $\tilde{A} = \tilde{A}^j \tau_j$.

Recall the discussion at the end of Section 2.3, $H_{ab} \in \text{SU}(2)$ can be diagonalized by

$$H_{ab} = M_{ab} \begin{pmatrix} x_{ab} & 0 \\ 0 & x_{ab}^{-1} \end{pmatrix} M_{ab}^{-1}, \quad M_{ab} = (\xi_{ab}, J\xi_{ab}). \quad (3.1)$$

The eigenvalue $x_{ab} \in \text{U}(1)$ relates to the area and constant curvature by $x_{ab} = \exp[-i\nu \frac{\Delta}{6} \mathbf{a}_{ab} + i\pi \varsigma_{ab}]$ where $\varsigma_{ab} \in \{0, 1\}$. $x_{ab} \neq 1$ is always assumed in our discussion by the nondegeneracy of geometry. Here the sign ambiguity $\nu = \pm 1$ corresponds to the Weyl reflection of $\text{SL}(2, \mathbb{C})$, which acts trivially on the character variety $\mathcal{M}_{\text{flat}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C}))$. Here the eigen-spinors $\xi_{ab}, J\xi_{ab} \in \mathbb{C}^2$ are set to be normalized with respect to the Hermitian inner product, and relates the unit 3-vector \hat{n}_{ab} by $\hat{n}_{ab} = \langle \xi, \vec{\sigma} \xi \rangle$. Given a 2-spinor $\xi = (\xi^1, \xi^2)^T$, $J\xi = (-\bar{\xi}^2, \bar{\xi}^1)^T$. $\langle \xi, \xi' \rangle = \bar{\xi}^1 \xi'^1 + \bar{\xi}^2 \xi'^2$ is the Hermitian inner product between 2-spinors.

Similarly, H_{ba} can be diagonalized and leads to the same eigenvalue x_{ab} as H_{ab} , but with the different eigen-spinors $\xi_{ba}, J\xi_{ba}$:

$$H_{ba} = M_{ba} \begin{pmatrix} x_{ab} & 0 \\ 0 & x_{ab}^{-1} \end{pmatrix} M_{ba}^{-1}, \quad M_{ba} = (\xi_{ba}, J\xi_{ba}). \quad (3.2)$$

where ξ_{ba} is again normalized by Hermitian inner product of 2-spinors.

By using the relation $H_{ab} = G_{ab} H_{ba} G_{ab}^{-1}$ and the diagonalizations Eqs.(3.1) and (3.2), we have

$$\begin{pmatrix} x_{ab} & 0 \\ 0 & x_{ab}^{-1} \end{pmatrix} M_{ab}^{-1} G_{ab} M_{ba} = M_{ab}^{-1} G_{ab} M_{ba} \begin{pmatrix} x_{ab} & 0 \\ 0 & x_{ab}^{-1} \end{pmatrix} \quad (3.3)$$

which implies that $M_{ab}^{-1} G_{ab} M_{ba}$ is a diagonal matrix¹⁷. We parametrize $M_{ab}^{-1} G_{ab} M_{ba}$ by

$$M_{ab}^{-1} G_{ab} M_{ba} = \begin{pmatrix} \lambda_{ab} & 0 \\ 0 & \lambda_{ab}^{-1} \end{pmatrix}. \quad (3.5)$$

Now we review briefly the definition of complex FN twist variable y_{ab} conjugate to x_{ab} , adapted to our context (see also e.g. [32]). For a meridian holonomy $H_{ab}(\mathfrak{p}_0)$ located at a generic base point \mathfrak{p}_0 on the cylinder connecting \mathcal{S}_a and \mathcal{S}_b (not necessarily located at \mathfrak{p}_a or \mathfrak{p}_b), we specify one of its eigenspace, denoted by d_{ab} , corresponding to the eigenvalue x_{ab} . d_{ab} may be viewed as a \mathbb{C}^2 -vector located at the base point \mathfrak{p}_0 , determined up to complex rescaling. d_{ab} breaks the \mathbb{Z}_2 symmetry from Weyl reflection, and is called a *framing flag* [32, 63, 64]¹⁸. Moreover a flat section s_{ab} over Σ_6 can be defined by the flatness equation (or parallel transportation)

$$(d - A)s_{ab} = 0 \quad (3.6)$$

with the initial value d_{ab} . s_{ab} at a point \mathfrak{p} is the eigenvector of $H_{ab}(\mathfrak{p})$ based at \mathfrak{p} . H_{ab} and H_{ba} give the same section $s_{ab} = s_{ba}$. s_{ab} at \mathfrak{p}_a and \mathfrak{p}_b are proportional to ξ_{ab} and ξ_{ba} respectively. Importantly the flat section s_{ab} cannot be made globally single-valued and smooth on Σ_6 . The monodromy H_{ab} requires to introduce a branch cut somewhere. s_{ab} is uniquely determined up to complex rescaling.

On the 4-holed sphere \mathcal{S}_a , in addition to the hole for gluing $\mathcal{S}_a, \mathcal{S}_b$, we pick two other holes corresponding to the meridian holonomies H_{ac} and H_{ae} . Similarly for \mathcal{S}_b , in addition to the hole for gluing $\mathcal{S}_a, \mathcal{S}_b$, we pick two other holes corresponding to the meridian holonomies H_{bd} and H_{bh} . The FN twist variable defined below depends on the choice of the 2 pair of holes. But the choice doesn't affect our following analysis. The

¹⁷If we write $M_{ab}^{-1} G_{ab} M_{ba} = \alpha^0 + \sigma_j \alpha^j$, then it implies that $\alpha^1 = \alpha^2 = 0$ by the linear-independence of Pauli matrices

$$0 = \left[\frac{x + x^{-1}}{2} + \frac{x - x^{-1}}{2} \sigma_3, \alpha^0 + \sigma_j \alpha^j \right] = \frac{x - x^{-1}}{2} [\sigma_3, \sigma_j] \alpha^j = i(x - x^{-1})(\alpha^1 \sigma^2 - \alpha^2 \sigma^1). \quad (3.4)$$

$x_{ab} \neq 1$ is always assumed in our discussion.

¹⁸A flat connection on Σ_g with a choices of framing flags is usually called a *framed* flat connection. All the flat connections considered in the present paper can be understood as the framed flat connections.

flat sections as eigenvectors of $H_{ab}, H_{ac}, H_{ae}, H_{bd}, H_{bh}$ give the following combination, which is independent of rescaling the sections:

$$\tau_{ab} = -\frac{\langle s_{bd} \wedge s_{bh} \rangle}{\langle s_{bd} \wedge s_{ab} \rangle \langle s_{bh} \wedge s_{ab} \rangle} \frac{\langle s_{ac} \wedge s_{ab} \rangle \langle s_{ae} \wedge s_{ab} \rangle}{\langle s_{ac} \wedge s_{ae} \rangle} \quad (3.7)$$

where $\langle v_1 \wedge v_2 \rangle \equiv \varepsilon_{\alpha\beta} v_1^\alpha v_2^\beta \forall v_1, v_2 \in \mathbb{C}^2$ is the $\text{SL}(2, \mathbb{C})$ invariant bilinear form on \mathbb{C}^2 . τ_{ab} is the complex FN twist variable for $\text{PSL}(2, \mathbb{C})$ flat connections, for being independent of the rescaling of sections s_{ab} . The $\text{SL}(2, \mathbb{C})$ FN twist variable y_{ab} is a lift of τ_{ab} by the relation $\tau_{ab} = -y_{ab}^2$. Since we would like to parametrizing $\text{SL}(2, \mathbb{C})$ flat connections, we choose a canonical lift from $\text{PSL}(2, \mathbb{C})$ data to define y_{ab} as the coordinate for $\text{SL}(2, \mathbb{C})$ flat connections.

Since τ_{ab} is invariant under the rescaling of sections, we fix all sections s_{ab} to be ξ_{ab} at \mathfrak{p}_a . By the $\text{SL}(2, \mathbb{C})$ invariance of $\langle v_1 \wedge v_2 \rangle$, we evaluate $\langle s_{bd} \wedge s_{bh} \rangle$ at \mathfrak{p}_b so that

$$\langle s_{bd} \wedge s_{bh} \rangle = \langle \xi_{bd} \wedge \xi_{bh} \rangle. \quad (3.8)$$

Similarly, we evaluate $\langle s_{ac} \wedge s_{ab} \rangle, \langle s_{ae} \wedge s_{ab} \rangle, \langle s_{ac} \wedge s_{ae} \rangle$ at \mathfrak{p}_a :

$$\frac{\langle s_{ac} \wedge s_{ab} \rangle \langle s_{ae} \wedge s_{ab} \rangle}{\langle s_{ac} \wedge s_{ae} \rangle} = \frac{\langle \xi_{ac} \wedge \xi_{ab} \rangle \langle \xi_{ae} \wedge \xi_{ab} \rangle}{\langle \xi_{ac} \wedge \xi_{ae} \rangle} \quad (3.9)$$

The factors $\langle s_{bd} \wedge s_{ab} \rangle$ and $\langle s_{bh} \wedge s_{ab} \rangle$ are evaluated conveniently at \mathfrak{p}_a , since s_{ab} has been fixed to be ξ_{ab} at \mathfrak{p}_a :

$$\begin{aligned} \langle s_{bd} \wedge s_{ab} \rangle &= \langle G_{ab} \xi_{bd} \wedge \xi_{ab} \rangle = \lambda_{ab}^{-1} \langle \xi_{bd} \wedge \xi_{ba} \rangle \\ \langle s_{bh} \wedge s_{ab} \rangle &= \langle G_{ab} \xi_{bh} \wedge \xi_{ab} \rangle = \lambda_{ab}^{-1} \langle \xi_{bh} \wedge \xi_{ba} \rangle \end{aligned} \quad (3.10)$$

where we have used Eq.(3.5) to show $G_{ab} \xi_{ba} = \lambda_{ab} \xi_{ab}$. As a result,

$$-\tau_{ab} = y_{ab}^2 = \lambda_{ab}^2 \chi_{ab}(\xi), \quad \chi_{ab}(\xi) = \frac{\langle \xi_{bd} \wedge \xi_{bh} \rangle}{\langle \xi_{bd} \wedge \xi_{ba} \rangle \langle \xi_{bh} \wedge \xi_{ba} \rangle} \frac{\langle \xi_{ac} \wedge \xi_{ab} \rangle \langle \xi_{ae} \wedge \xi_{ab} \rangle}{\langle \xi_{ac} \wedge \xi_{ae} \rangle}. \quad (3.11)$$

Given an flat connection $\mathfrak{A} \in \mathcal{M}_{\text{flat}}(\Sigma_6, \text{SL}(2, \mathbb{C}))$ satisfying the boundary condition on Σ_6 , which can be extended to be an $\text{SL}(2, \mathbb{C})$ flat connection A on $S^3 \setminus \Gamma_5$, the previous analysis has given us the relation between the longitude holonomy G_{ab} of A and the hyperdihedral angles Θ_{ab} of the constant curvature 4-simplex. Moreover Eq.(3.5) coincides with the parallel transportation equations in [20]. It is not hard to see that y_{ab} relates the 4d dihedral angles Θ_{ab} , which is the boost parameter relating the local frames $e_I(a)$ and $e_I(b)$ of tetra_a and tetra_b . More precisely

$$-\tau_{ab} = y_{ab}^2 = e^{-\nu \text{sgn}(V_4) \Theta_{ab} - 2i\nu \theta_{ab}} \chi_{ab}(\xi) \quad (3.12)$$

which relates the complex FN twist coordinate of flat connection on Σ_6 (which is extendable to $S^3 \setminus \Gamma_5$) to the dihedral angle of constant curvature 4-simplex. $\nu = \text{sgn}(\Lambda)$ and $\text{sgn}(V_4) = \pm 1$ is the orientation of the geometrical 4-simplex. In Appendix D, we provide a proof of the above relation in order to be self-contained, see also [20].

We make a lift from $\text{PSL}(2, \mathbb{C})$ FN twist τ_{ab} to $\text{SL}(2, \mathbb{C})$ FN twist y_{ab} . We have

$$y_{ab} = \pm e^{-\frac{1}{2}\nu \text{sgn}(V_4) \Theta_{ab} - i\nu \theta_{ab}} \sqrt{\chi_{ab}(\xi)} \quad (3.13)$$

Here \pm labels the 2 branches of $\text{SL}(2, \mathbb{C})$ flat connections which are identified when they are projected to $\text{PSL}(2, \mathbb{C})$ flat connections. Here we prefer to use $\text{SL}(2, \mathbb{C})$ coordinates to parametrize the 4-simplex geometry. In order to use $\text{SL}(2, \mathbb{C})$ FN twist y_{ab} for the parametrization, we make a canonical choice of lift, e.g. the plus sign in the above (minus sign works as well, the only importance is to make y_{ab} single-valued).

We write $x_{ab} = e^{u_{ab}}$ and $y_{ab} = e^{-\frac{2\pi}{t}v_{ab}}$, the holomorphic Chern-Simons (Atiyah-Bott-Goldman) symplectic form $\omega_{CS} = \frac{t}{4\pi} \int_{\Sigma_{g=6}} \text{tr} [\delta_1 A \wedge \delta_2 A]$ gives the Poisson bracket $\{A_\mu^i(x), A_\nu^j(x')\} = (-\frac{8\pi}{t}) \epsilon_{\mu\nu} \delta^{ij} \delta^{(2)}(x, x')$, which implies

$$\{u_{ab}, v_{a'b'}\} = \delta_{(ab), (a'b')}. \quad (3.14)$$

The derivation of Poisson bracket between u_{ab} and v_{ab} can be found in Appendix C.

The logarithmic coordinate v_{ab} relates to the hyperdihedral angles Θ_{ab} by

$$-\frac{2\pi}{t}v_{ab} = -\frac{1}{2}v \text{sgn}(V_4) \Theta_{ab} - iv \theta_{ab} + \ln \sqrt{\chi_{ab}(\xi)} + 2\pi N_{ab}i, \quad N_{ab} \in \mathbb{Z}, \quad (3.15)$$

The parity partner flat connection \tilde{A} gives the twist variable $\tilde{y}_{ab} = e^{-\frac{2\pi}{\tilde{t}}\tilde{v}_{ab}}$. The parity transformation $A \mapsto \tilde{A}$ doesn't affect $\chi_{ab}(\xi)$ and θ_{ab} , but flip the sign of $\text{sgn}(V_4)$. Thus \tilde{v}_{ab} is written as

$$-\frac{2\pi}{\tilde{t}}\tilde{v}_{ab} = -\frac{1}{2}\tilde{v} \text{sgn}(\tilde{V}_4) \Theta_{ab} - i\tilde{v} \theta_{ab} + \ln \sqrt{\chi_{ab}(\xi)} + 2\pi \tilde{N}_{ab}i, \quad \tilde{N}_{ab} \in \mathbb{Z}, \quad (3.16)$$

where $\text{sgn}(\tilde{V}_4) = -\text{sgn}(V_4)$. Except the discrete N_{ab} ambiguities, v_{ab}, \tilde{v}_{ab} are only different by a sign flip in front of Θ_{ab} at the first term. In the above definition of twist variable, the choice of 2 pairs of holes on S_a and S_b only affects $\ln \chi_{ab}(\xi)$, which however doesn't play any role in the following analysis¹⁹.

4 Quantization of 3d Flat Connection v.s. Quantum 4d Geometry/Gravity

4.1 $\text{SL}(2, \mathbb{C})$ Chern-Simons Theory and holomorphic 3d block

We have established above the correspondence between the $\text{SL}(2, \mathbb{C})$ flat connection on 3-manifold $M_3 = S^3 \setminus \Gamma_5$ and the 4-geometry of a constant curvature 4-simplex. Such a correspondence provides us a way to quantize 4-dimensional simplicial geometry via the quantization of flat connection on 3-manifold. The resulting quantum geometry gives a model for simplicial quantum gravity in 4 dimensions.

A well-studied quantization of flat connection on 3-manifold is given by Chern-Simons theory. In the following we study the $\text{SL}(2, \mathbb{C})$ Chern-Simons theory on $M_3 = S^3 \setminus \Gamma_5$ with the implementation of boundary condition proposed in Section 2.2. We show that the resulting Chern-Simons theory is a good candidate of the quantum theory of 4-dimensional simplicial geometry/gravity, in the sense that the theory reproduces the classical action of 4-dimensional Einstein gravity in the semiclassical regime.

The $\text{SL}(2, \mathbb{C})$ Chern-Simons theory on $M_3 = S^3 \setminus \Gamma_5$ is defined by the following action [22]:

$$\begin{aligned} CS[M_3 | A, \bar{A}] &= \frac{t}{8\pi} \int_{M_3} \text{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) + \frac{\bar{t}}{8\pi} \int_{M_3} \text{tr} \left(\bar{A} \wedge d\bar{A} + \frac{2}{3} \bar{A} \wedge \bar{A} \wedge \bar{A} \right) \\ &+ \frac{t}{8\pi} \int_{\partial M_3} \text{tr} (A_1 \wedge A_2) + \frac{\bar{t}}{8\pi} \int_{\partial M_3} \text{tr} (\bar{A}_1 \wedge \bar{A}_2). \end{aligned} \quad (4.1)$$

Here the Chern-Simons couplings $t = k + is$ and $\bar{t} = k - is$ are assumed to be complex conjugate to each other with $k, s \in \mathbb{R}$. When $k \in \mathbb{Z}$, $\exp[iCS]$ is invariant under large gauge transformation. However in most of the following discussion we keep k to be arbitrary real number without restricting $k \in \mathbb{Z}$ [6]. A coordinate s_1, s_2 has been chosen on the boundary $\partial M_3 = \Sigma_6$, such that e.g. s_1 is the meridian direction of Γ_5 . In the boundary terms of the action, A_1, A_2 are the components of connection A along the direction s_1, s_2 . By the boundary terms, the boundary values of A_1, \bar{A}_1 should be specified to define the variational principle [65]²⁰.

¹⁹ $\chi_{ab}(\xi)$ is Poisson commuting with u_{ab} , because the holonomies involved in $\chi_{ab}(\xi)$ have vanishing intersection number with a meridian holonomy on the cylinder connecting S_a and S_b (see the computation in Appendix C). Therefore one can even make a canonical transformation and remove the term $\frac{\ln \chi_{ab}(\xi)}{2}$ in $-\frac{2\pi}{t}v_{ab}$.

²⁰The boundary term from the variational principle of $CS[M_3 | A, \bar{A}]$ is $\frac{t}{4\pi} \int_{\partial M_3} \text{tr} (\delta A_1 \wedge A_2) + \frac{\bar{t}}{4\pi} \int_{\partial M_3} \text{tr} (\delta \bar{A}_1 \wedge \bar{A}_2) = 0$ when the boundary values of A_1, \bar{A}_1 is specified.

The functional integration quantization defines the Chern-Simons partition function on $S^3 \setminus \Gamma_5$ by

$$Z_{CS}(S^3 \setminus \Gamma_5 | A_1, \bar{A}_1) = \int_{A_1, \bar{A}_1} \mathcal{D}A \mathcal{D}\bar{A} e^{\frac{i}{\hbar} CS[S^3 \setminus \Gamma_5 | A, \bar{A}]} \quad (4.2)$$

A_1, \bar{A}_1 set up the boundary condition on Σ_6 for the functional integral on $S^3 \setminus \Gamma_5$, while A_2, \bar{A}_2 on the boundary are integrated in the functional integral. $1/\hbar \in \mathbb{R}$ may be viewed as a scaling parameter for t, \bar{t} . The semiclassical limit $\hbar \rightarrow 0$ is the same as scaling t, \bar{t} uniformly to infinity.

$Z_{CS}(S^3 \setminus \Gamma_5 | A_1, \bar{A}_1)$ is a “wave function” of Chern-Simons theory, i.e. a (possibly distributional) state in the Hilbert space $\mathcal{H}(\Sigma_6)$ defined on the boundary Σ_6 . The Hilbert space $\mathcal{H}(\Sigma_6)$ is a quantization of $\mathcal{M}_{\text{flat}}(\Sigma_6, \text{SL}(2, \mathbb{C}))$ the moduli space of $\text{SL}(2, \mathbb{C})$ flat connections on the genus-6 closed 2-surface Σ_6 [7, 8, 24, 32, 66, 67]. The moduli space of flat connection $\mathcal{M}_{\text{flat}}(\Sigma_g, \text{SL}(2, \mathbb{C}))$ on a genus- g closed 2-surface Σ_g is a hyper-kähler variety of $\dim_{\mathbb{C}} = 6g - 6$, and is known as the Hitchin moduli space [31]. The closed 2-surface Σ_g can be decomposed into pairs of pants by cutting through $3g - 3$ closed curves $\{c_m\}_{m=1}^{3g-3}$ as e.g. in Fig. 6, such that $\Sigma_g \setminus \{c_m\}_{m=1}^{3g-3}$ is a set of 3-holed spheres (pairs of pants). A flat connection on Σ_g gives a set of holonomies H_m along the closed curves $\{c_m\}_{m=1}^{3g-3}$. The set of holonomy eigenvalues $\{x_m\}_{m=1}^{3g-3}$ and their canonical conjugate variables $\{y_m\}_{m=1}^{3g-3}$ define the complex FN coordinate on $\mathcal{M}_{\text{flat}}(\Sigma_g, \text{SL}(2, \mathbb{C}))$. As discussed above in the case $\Sigma_6 = \partial(S^3 \setminus \Gamma_5)$ with the boundary condition, some of the complex FN variables relate the triangle areas a_{ab} and hyperdihedral angles Θ_{ab} in the (constant curvature) simplicial geometry.

The Chern-Simons symplectic structure on $\mathcal{M}_{\text{flat}}(\Sigma_g, \text{SL}(2, \mathbb{C}))$ can be locally expressed in terms of the FN coordinates by

$$\omega_{CS} = \left(-\frac{t}{2\pi}\right) \sum_{m=1}^{3g-3} \frac{dy_m}{y_m} \wedge \frac{dx_m}{x_m} + c.c. = \sum_{m=1}^{3g-3} dv_m \wedge du_m + c.c. \quad (4.3)$$

with

$$x_m = e^{u_m}, \quad y_m = e^{-\frac{2\pi}{t} v_m}, \quad \bar{x}_m = e^{\bar{u}_m}, \quad \bar{y}_m = e^{-\frac{2\pi}{\bar{t}} \bar{v}_m}. \quad (4.4)$$

The FN coordinate can be quantized to be \hat{u}_m, \hat{v}_m and $\hat{\bar{u}}_m, \hat{\bar{v}}_m$ with their canonical commutation relations

$$[\hat{u}_m, \hat{v}_n] = i\hbar \delta_{mn}, \quad [\hat{\bar{u}}_m, \hat{\bar{v}}_n] = i\hbar \delta_{mn}, \quad (4.5)$$

or in terms of \hat{x}_m and \hat{y}_m

$$\hat{x}_m \hat{y}_m = e^{-\frac{2\pi i \hbar}{t}} \hat{y}_m \hat{x}_m \quad \text{and} \quad \hat{x}_n \hat{y}_m = \hat{y}_m \hat{x}_n \quad \text{when } n \neq m. \quad (4.6)$$

and similarly for $\hat{\bar{x}}_m$ and $\hat{\bar{y}}_m$.

The Hilbert space $\mathcal{H}(\Sigma_g)$ (of L^2 -type) is the representation of the above canonical commutation relations. A state in $\mathcal{H}(\Sigma_g)$ is a wave function of u, \bar{u} , e.g. the path integral Eq.(4.2) should be written as $Z_{CS}(S^3 \setminus \Gamma_5 | u, \bar{u})$. $\hat{x}, \hat{\bar{x}}$ or $\hat{u}, \hat{\bar{u}}$ are represented as the multiplication operators, while $\hat{v}_m = -i\hbar \partial_{u_m}$ and $\hat{\bar{v}}_m = -i\hbar \partial_{\bar{u}_m}$ are represented as the derivatives.

The solutions of Chern-Simons equation of motion $F(A) = F(\bar{A}) = 0$ define the moduli space of flat connections $\mathcal{M}_{\text{flat}}(M_3, \text{SL}(2, \mathbb{C}))$ on a 3-manifold. The moduli space $\mathcal{M}_{\text{flat}}(M_3, \text{SL}(2, \mathbb{C})) = \mathcal{L}_{\mathbf{A}}$ is a holomorphic Lagrangian subvariety in $\mathcal{M}_{\text{flat}}(\partial M_3 = \Sigma_g, \text{SL}(2, \mathbb{C}))$ [32, 33]. $M_3 = S^3 \setminus \Gamma_5$ and $\partial M_3 = \Sigma_6$ in our case. $\mathcal{L}_{\mathbf{A}}$ can be represented at least locally in $\mathcal{M}_{\text{flat}}(\Sigma_6, \text{SL}(2, \mathbb{C}))$ by a set of (Laurent) polynomial equations $\mathbf{A}_m(x, y) = 0$ with $m = 1, \dots, 3g - 3$. In quantum Chern-Simons theory, the Lagrangian subvariety $\mathcal{L}_{\mathbf{A}}$ are quantized via the quantization of the polynomial equations $\mathbf{A}_m(x, y) = 0$, which gives the holomorphic operator constraint equations:

$$\hat{\mathbf{A}}_m(\hat{x}, \hat{y}, \hbar) f(u) = 0, \quad m = 1, \dots, 3g - 3 \quad (4.7)$$

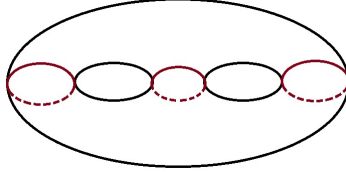


Figure 6. The pants decomposition of a genus-2 closed 2-surface by cutting through the red curves.

where $\hat{\mathbf{A}}_m(\hat{x}, \hat{y}, \hbar)$ is the quantization of $\mathbf{A}_m(x, y)$ with a certain operator ordering [68]. $f(u)$ is the physical wave function of $\text{SL}(2, \mathbb{C})$ Chern-Simons theory, which is holomorphic by the holomorphicity of \mathcal{L}_A or $\mathbf{A}_m(x, y) = 0$.

In particular, the functional integration $Z_{CS}(S^3 \setminus \Gamma_5 | u, \bar{u})$ in Eq.(4.2), satisfies the above operator constraint equation and its complex conjugate:

$$\hat{\mathbf{A}}_m(\hat{x}, \hat{y}, \hbar) Z_{CS}(M_3 | u, \bar{u}) = 0, \quad \hat{\mathbf{A}}_m(\hat{\bar{x}}, \hat{\bar{y}}, \hbar) Z_{CS}(M_3 | u, \bar{u}) = 0. \quad (4.8)$$

It has been shown in [6, 25] that $Z_{CS}(M_3 | u, \bar{u})$ can be written in the following form:

$$Z_{CS}(M_3 | u, \bar{u}) = \sum_{\alpha, \bar{\alpha}} n_{\alpha, \bar{\alpha}} Z_{CS}^{(\alpha)}(M_3 | u) Z_{CS}^{(\bar{\alpha})}(M_3 | \bar{u}). \quad (4.9)$$

$Z_{CS}^{(\alpha)}(M_3 | u)$ is called the “holomorphic 3d block” satisfying the holomorphic operator constraint equation:

$$\hat{\mathbf{A}}_m(\hat{x}, \hat{y}, \hbar) Z_{CS}^{(\alpha)}(M_3 | u) = 0, \quad \hat{\mathbf{A}}_m(\hat{\bar{x}}, \hat{\bar{y}}, \hbar) Z_{CS}^{(\bar{\alpha})}(M_3 | \bar{u}) = 0, \quad \forall \alpha, \bar{\alpha}. \quad (4.10)$$

The holomorphic 3d block $Z_{CS}^{(\alpha)}(M_3 | u)$ is the central object to be studied in the following. $Z_{CS}^{(\alpha)}(M_3 | u)$ is a path integral of holomorphic $\text{SL}(2, \mathbb{C})$ Chern-Simons theory (holomorphic part of Eq.(4.1)), defined on a certain integration cycle passing through a single saddle point (being a flat connection). The integration cycle defining each $Z_{CS}^{(\alpha)}(M_3 | u)$ is one of the Lefschetz thimbles from Chern-Simons path integral [6].

Each holomorphic 3d block, written as an asymptotic expansion in \hbar , can also be understood from the WKB analysis of the above operator constraint equations.

$$Z_{CS}^{(\alpha)}(M_3 | u) = \exp \left[\frac{i}{\hbar} \int_{\mathfrak{C} \subset \mathcal{L}_A}^{(u, v^{(\alpha)})} \vartheta + \dots \right], \quad Z_{CS}^{(\bar{\alpha})}(M_3 | \bar{u}) = \exp \left[\frac{i}{\hbar} \int_{\mathfrak{C} \subset \mathcal{L}_A}^{(\bar{u}, \bar{v}^{(\bar{\alpha})})} \bar{\vartheta} + \dots \right]. \quad (4.11)$$

ϑ and $\bar{\vartheta}$ are the holomorphic and anti-holomorphic part of the Liouville 1-form, which can be written locally in terms of FN coordinate:

$$\vartheta := \left(-\frac{t}{2\pi} \right) \sum_{m=1}^{3g-3} \ln y_m \frac{dx_m}{x_m}, \quad \bar{\vartheta} := \left(-\frac{\bar{t}}{2\pi} \right) \sum_{m=1}^{3g-3} \ln \bar{y}_m \frac{d\bar{x}_m}{\bar{x}_m} \quad (4.12)$$

α (or $\bar{\alpha}$) labels the branch of the Lagrangian subvariety \mathcal{L}_A , on which $v_m^{(\alpha)}$ are single-valued with respect to u_m from solving $\mathbf{A}_m(x, y) = 0$. The integral in Eq.(4.11) is along a contour \mathfrak{C} in the Lagrangian subvariety \mathcal{L}_A connecting the flat connection $(u, v^{(\alpha)})$ in branch α and a reference flat connection (u_0, v_0) . In our context, both flat connections at the end points of \mathfrak{C} in Eq.(4.11) are covered in a FN coordinate chart. “...” contains the subleading terms of $\log \hbar$ and $\sum_{n=0}^{\infty} S_n^{(\alpha)}(u) \hbar^{n21}$. The computation of coefficients $n_{\alpha\bar{\alpha}}$ in Eq.(4.9) is described in [6]. The above discussion provides a perturbative definition of holomorphic 3d block $Z_{CS}^{(\alpha)}(M_3 | u)$. $Z_{CS}^{(\alpha)}(M_3 | u)$ also has a nonperturbative definition as a “state-integral model” [8, 25, 41].

²¹All the quantum corrections $S_n(u)^{(\alpha)}$ can be computed recursively [25], and relate to the technique of topological recursion at least when M_3 is a knot complement [70].

When $\text{Re}(t) = k \in \mathbb{Z}$ and $\hbar^{-1} \in \mathbb{Z}$ (\hbar^{-1} is understood simply as a scaling parameter for t, \bar{t}), the Lagrangian subvariety \mathcal{L}_A is required to be quantizable, namely the integrals in Eq.(4.11) don't depend on the choice of the contour by $\oint v \cdot du = 2\pi\hbar\mathbb{Z}$ on \mathcal{L}_A . Such a requirement indeed holds, which has a beautiful algebraic K -theory interpretation: \mathcal{L}_A is Lagrangian in a stronger sense, i.e. it is a K_2 -Lagrangian subvariety [33, 68, 71]. A very brief explanation is given in the Appendix E. In the case of knot complement 3-manifold (3-manifold with torus cusps), \mathcal{L}_A being quantizable is understood much earlier in [72–75].

The freedom of the overall phase in the wave function $Z_{CS}(M_3|u, \bar{u})$ relates to the choice of the reference flat connection $(u_0, v_0), (\bar{u}_0, \bar{v}_0)$. Let u, \bar{u} are the boundary values defining the path integral $Z_{CS}(M_3|u, \bar{u})$, we can choose the reference flat connection to be a pair of solutions $(u, v^{(\alpha_0)}), (\bar{u}, \bar{v}^{(\bar{\alpha}_0)})$ of $\mathbf{A}_m(u, v) = 0$, where $\alpha_0, \bar{\alpha}_0$ denote the reference branches. Then the phase difference between another pair of flat connections $(u, v^{(\alpha)}), (\bar{u}, \bar{v}^{(\bar{\alpha})})$ in the branches $\alpha, \bar{\alpha}$ and the reference pair $(u, v^{(\alpha_0)}), (\bar{u}, \bar{v}^{(\bar{\alpha}_0)})$ is given by

$$Z_{CS}^{(\alpha)}(M_3|u) = \exp \left[\frac{i}{\hbar} \int_{\substack{(u, v^{(\alpha_0)}) \\ \in \mathcal{L}_A}}^{(u, v^{(\alpha)})} \vartheta + \dots \right], \quad Z_{CS}^{(\bar{\alpha})}(M_3|\bar{u}) = \exp \left[\frac{i}{\hbar} \int_{\substack{(\bar{u}, \bar{v}^{(\bar{\alpha}_0)}) \\ \in \mathcal{L}_A}}^{(\bar{u}, \bar{v}^{(\bar{\alpha})})} \bar{\vartheta} + \dots \right]. \quad (4.13)$$

Although the Lagrangian subvariety \mathcal{L}_A is defined by $\mathbf{A}_m(x, y)$ being a function of x_m, y_m , the holomorphic 3d block $Z_{CS}^{(\alpha)}(M_3|u)$ is rather a function of the logarithmic coordinates u_m , satisfying Eq.(4.10). In other words, $Z_{CS}^{(\alpha)}(M_3|u)$ is in general not a periodic function of u under $u \rightarrow u + 2\pi i$. Similarly $Z_{CS}^{(\alpha)}(M_3|u)$ and $Z_{CS}^{(\alpha')} (M_3|u)$ are 2 different 3d holomorphic blocks even when $v^{(\alpha)}$ and $v^{(\alpha')}$ give the same $y_m = e^{-\frac{2\pi}{i}v_m}$. The reason is essentially that $Z_{CS}^{(\alpha)}(M_3|u)$ is defined by the path integral of an analytic continuation of Chern-Simons theory with t extended to an arbitrary complex number (see [6] and the 2nd reference in [8]). The requirement $k \in \mathbb{Z}$ is relaxed so that $Z_{CS}^{(\alpha)}(M_3|u)$ as a path integral is not defined on the space of gauge equivalent classes of connections but rather the universal cover space, without quotient out the large gauge transformations. Therefore in general $Z_{CS}^{(\alpha)}(M_3|u) \neq Z_{CS}^{(\alpha')} (M_3|u')$, even when $(u, v^{(\alpha)})$ and $(u', v^{(\alpha')})$ correspond to the same flat connection (x, y) on M_3 . The integration contour \mathfrak{C} essentially lies in the cover space of \mathcal{L}_A . In analytic continued Chern-Simons theory, we have $\oint v \cdot du = 0$ on the cover space of \mathcal{L}_A , which is explained in Appendix E.

Let's fix u , and consider $(u, v^{(\alpha)})$ and $(u, v^{(\alpha')})$ corresponding to the same flat connection (x, y) on M_3 , where $v^{(\alpha)}, v^{(\alpha')}$ corresponding to different lifts of $y = e^{-\frac{2\pi}{i}v}$ to the cover space. $v^{(\alpha)}, v^{(\alpha')}$ is different by an integer multiple of it . Thus the classical term in Eq.(4.11) $\int^{(u, v^{(\alpha)})} \vartheta$ and $\int^{(u, v^{(\alpha')})} \vartheta$ is different by integer multiple of $it u_m$. There is no difference in the quantum corrections between $S_n^{(\alpha)}(u)$ and $S_n^{(\alpha')}(u)$ [25].

There is one more remark on the holomorphic 3d blocks: $Z_{CS}^{(\alpha)}(M_3|u)$ comes from the holomorphic quantization and satisfies the holomorphic operator constraint equation Eq.(4.10). It is natural to consider such a class of holomorphic functions by the holomorphic nature of the Lagrangian subvariety \mathcal{L}_A . The class of holomorphic 3d blocks provides a basis in the space of solutions of the operator constraint equation Eq.(4.10). Moreover such a holomorphic quantization has the interesting interpretation in terms of ‘‘Brane quantization’’ [12]. Rigorously speaking, the holomorphic wave functions $Z_{CS}^{(\alpha)}(M_3|u)$ doesn't live in the L^2 Hilbert space $\mathcal{H}(\Sigma_g)$, which is a typical phenomena of solving operator constraint equation. The holomorphic 3d block $Z_{CS}^{(\alpha)}(M_3|u)$, which is the central object in the following investigation, contains rich enough structures in both mathematics and physics (see e.g. [8, 25, 76, 78]).

4.2 Asymptotics of holomorphic 3d Block and Simplicial Quantum Gravity

In order to obtain the quantization of flat connection as a quantum theory of simplicial 4-geometry, the boundary condition described in Section 2.2 should be imposed to the $\text{SL}(2, \mathbb{C})$ Chern-Simons theory on the graph complement $S^3 \setminus \Gamma_5$. The subspace of flat connections satisfying the boundary condition is denoted by $\mathcal{N}^{b.c.} \in \mathcal{M}_{\text{flat}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C}))$. As discussed previously, the flat connections in $\mathcal{N}^{b.c.}$ can be parametrized

by the complex FN variables $[x_{ab}; x_a, y_a]$ satisfying (1) the complex FN length variable $x_{ab} \in U(1)$ for each pant curve c_{ab} , and (2) the length and twist variables x_a, y_a for each pant curve c_a parametrize an $SU(2)$ flat connection on 4-holed sphere \mathcal{S}_a with a given conjugacy classes x_{ab} for each hole. The boundary condition selects a subclass of 3d holomorphic blocks among all the ones satisfying Eq.(4.10). By the geometrical correspondence of flat connection on $S^3 \setminus \Gamma_5$, each holomorphic block in the subclass associates uniquely a geometrical 4-simplex with constant curvature, up to the possible degenerate geometry. Here we view that the holomorphic 3d block $Z_{CS}^{(\alpha)}(M_3|u)$ is a quantum state describing the quantum fluctuation of the corresponding 4-simplex geometry. In the following we analyze the asymptotic behavior of $Z_{CS}^{(\alpha)}(M_3|u)$ as $\hbar \rightarrow 0$. It turns out that the leading asymptotic behavior gives a 4d simplicial Einstein-Hilbert action on the exponential, which suggests that $Z_{CS}^{(\alpha)}(M_3|u)$ is actually a wave function of simplicial quantum gravity in 4 dimensions.

Let's consider a set of boundary data $[x_{ab}; x_a, y_a]$ which corresponds to the geometrical boundary of a constant curvature 4-simplex, and is extendible to a flat connection on $S^3 \setminus \Gamma_5$. Theorem 2.5 states that there are exactly two $SL(2, \mathbb{C})$ flat connections A, \tilde{A} on $S^3 \setminus \Gamma_5$, which are consistent with the boundary data. A, \tilde{A} on $S^3 \setminus \Gamma_5$ have two different boundary values $\mathfrak{A}, \tilde{\mathfrak{A}}$ as flat connections on Σ_6 . Both $\mathfrak{A}, \tilde{\mathfrak{A}}$ are covered by the complex FN coordinate chart: $\mathfrak{A} = [x_{ab}, y_{ab}; x_a, y_a]$ and $\tilde{\mathfrak{A}} = [x_{ab}, \tilde{y}_{ab}; x_a, y_a]$, which have 10 different twist variables $y_{ab} \neq \tilde{y}_{ab}$. By the geometrical interpretation of the FN variables, $x_{ab} = e^{u_{ab}}$ and $y_{ab} = e^{-\frac{2\pi}{t}v_{ab}}, \tilde{y}_{ab} = e^{-\frac{2\pi}{t}\tilde{v}_{ab}}$ relate 10 triangle areas and 10 hyperdihedral angles via

$$\begin{aligned} u_{ab} &= -iv\frac{\Lambda}{6}\mathbf{a}_{ab} + i\pi\mathfrak{s}_{ab} \bmod 2\pi i\mathbb{Z}, \\ v_{ab}^{(\alpha)} &= \frac{t}{4\pi}v\Theta_{ab} + \frac{it}{2\pi}v\theta_{ab} - \frac{t\ln\chi_{ab}(\xi)}{4\pi} - itvN_{ab}, \\ \tilde{v}_{ab}^{(\tilde{\alpha})} &= -\frac{t}{4\pi}v\Theta_{ab} + \frac{it}{2\pi}v\theta_{ab} - \frac{t\ln\chi_{ab}(\xi)}{4\pi} - itv\tilde{N}_{ab} \end{aligned} \quad (4.14)$$

where $\mathfrak{s}_{ab} \in \{0, 1\}$ and $N_{ab}, \tilde{N}_{ab} \in \mathbb{Z}$. In addition, x_a, y_a parametrize the shape of a constant curvature tetrahedron with a given set of triangle areas. As it has been discussed, we have to lift the moduli space of flat connection to its cover space and consider the logarithmic coordinates u and v , in order to define the holomorphic 3d block. We choose an (arbitrary) canonical lift for the boundary data x_{ab}, x_a, y_a to the logarithmic data u_{ab}, u_a, v_a . In Eq.(4.14), we have also chosen arbitrarily 2 lifts α and $\tilde{\alpha}$ for y_{ab} and \tilde{y}_{ab} . The difference between $v_{ab}^{(\alpha)}$ and $\tilde{v}_{ab}^{(\tilde{\alpha})}$ is given by

$$v_{ab}^{(\alpha)} - \tilde{v}_{ab}^{(\tilde{\alpha})} = \frac{t}{2\pi}v[\Theta_{ab} + 2\pi i\Delta N_{ab}], \quad (4.15)$$

where $\Delta N_{ab} = \tilde{N}_{ab} - N_{ab} \in \mathbb{Z}$.

With the above arbitrarily chosen lifts, the boundary data $[x_{ab}; x_a, y_a]$ (with the canonical lift) picks up two holomorphic 3d blocks $Z_{CS}^{(\alpha)}(M_3|u)$ and $Z_{CS}^{(\tilde{\alpha})}(M_3|u)$ corresponding respectively to the two $SL(2, \mathbb{C})$ flat connections A, \tilde{A} on $S^3 \setminus \Gamma_5$. To eliminate the overall phase ambiguity of the wave function, we consider $Z_{CS}^{(\alpha)}(M_3|u)$ of $(u, v^{(\alpha)})$ defined with respect to the reference $(u, v^{(\tilde{\alpha})})$. Namely we compute the phase difference between $Z_{CS}^{(\alpha)}(M_3|u)$ and $Z_{CS}^{(\tilde{\alpha})}(M_3|u)$. By Eq.(4.13),

$$Z_{CS}^{(\alpha)}(M_3|u) = \exp \left[\frac{i}{\hbar} \int_{\mathfrak{L}_{\mathcal{A}}}^{(u, v^{(\alpha)})} \vartheta + \dots \right] \quad (4.16)$$

Here the Liouville 1-form ϑ is given by

$$\vartheta = \sum_{a < b} v_{ab} du_{ab} + \sum_{a=1}^5 v_a du_a. \quad (4.17)$$

In the following we compute the integral $\int_{\mathfrak{C} \subset \mathcal{L}_A} \vartheta$ as the leading contribution in \hbar of the holomorphic 3d block.

The flat connections A and \tilde{A} correspond to the intersections $[x_{ab}, y_{ab}; x_a, y_a]$ and $[x_{ab}, \tilde{y}_{ab}; x_a, y_a]$ between the Lagrangian subvariety \mathcal{L}_A and the plane $\mathcal{P}_{[x_{ab}; x_a, y_a]}$ with constant $[x_{ab}; x_a, y_a]$. Theorem 2.5 can be rephrased that the Lagrangian subvariety \mathcal{L}_A and the plane $\mathcal{P}_{[x_{ab}; x_a, y_a]}$ intersect precisely at two different points, when $[x_{ab}; x_a, y_a]$ satisfies the boundary condition and corresponds geometrically the boundary of a constant curvature 4-simplex. The two intersection points are $[x_{ab}, y_{ab}; x_a, y_a]$ and $[x_{ab}, \tilde{y}_{ab}; x_a, y_a]$ corresponding to two different constant curvature 4-simplex related by a parity inversion.

We now consider any smooth variation of the boundary data $[x_{ab}(\eta); x_a(\eta), y_a(\eta)]$ from the original one $[x_{ab}; x_a, y_a]$, parametrized by a real parameter η , such that $[x_{ab}(\eta); x_a(\eta), y_a(\eta)]$ at a given η still satisfies the boundary condition and corresponds geometrically the boundary of a constant curvature 4-simplex. Thus we obtain a 1-parameter family of planes $\mathcal{P}_{[x_{ab}; x_a, y_a]}(\eta)$. Each plane in the family intersect \mathcal{L}_A at 2 different points. Each of intersection point corresponds to a constant curvature 4-simplex deviating smoothly from the original one $[x_{ab}, y_{ab}; x_a, y_a]$ or $[x_{ab}, \tilde{y}_{ab}; x_a, y_a]$. For example, a smooth variation of the edge-lengths of the constant curvature 4-simplex induces such a variation of $[x_{ab}(\eta); x_a(\eta), y_a(\eta)]$.

The smooth variation of boundary data $[x_{ab}(\eta); x_a(\eta), y_a(\eta)]$ leads to the smooth variation of their canonical lifts $[u_{ab}(\eta); u_a(\eta), v_a(\eta)]$. The variation being smooth doesn't change the branches or lifts. Similarly for the lifts of A and \tilde{A} , the branches α and $\tilde{\alpha}$ don't change under the smooth variation. The variations of $v_{ab}^{(\alpha)}$ and $\tilde{v}_{ab}^{(\tilde{\alpha})}$ give $v_{ab}^{(\alpha)}(\eta)$ and $\tilde{v}_{ab}^{(\tilde{\alpha})}(\eta)$. We define:

$$I_{\tilde{\alpha}}^{\alpha}(\eta) := \int_{\substack{(u(\eta), v^{(\alpha)}(\eta)) \\ \mathfrak{C}(\eta) \subset \mathcal{L}_A}}^{\substack{(u(\eta), v^{(\tilde{\alpha})}(\eta))}} \vartheta, \quad \text{and} \quad \delta_{\eta} I_{\tilde{\alpha}}^{\alpha}(\eta) = I_{\tilde{\alpha}}^{\alpha}(\eta + \delta\eta) - I_{\tilde{\alpha}}^{\alpha}(\eta) \quad (4.18)$$

By using the freedom of choosing the integration contour $\mathfrak{C}(\eta)$, we can make the contour $\mathfrak{C}(\eta + \delta\eta)$ include $\mathfrak{C}(\eta)$. Indeed, given a integration contour $\mathfrak{C}(\eta_0)$ connecting the pair of intersection points (in the cover space) $[u_{ab}(\eta_0), v_{ab}^{(\alpha)}(\eta_0); u_a(\eta_0), v_a(\eta_0)]$ and $[u_{ab}(\eta_0), \tilde{v}_{ab}^{(\tilde{\alpha})}(\eta_0); u_a(\eta_0), v_a(\eta_0)]$ for a fixed η_0 , we extend the contour from both intersection points by adding the curves $c = [u_{ab}(\eta), v_{ab}^{(\alpha)}(\eta); u_a(\eta), v_a(\eta)]$ and $\tilde{c}^{-1} = [u_{ab}(\eta), \tilde{v}_{ab}^{(\tilde{\alpha})}(\eta); u_a(\eta), v_a(\eta)]$, where the variational parameter η is viewed as the parameter of the curve extension. The extended contour still lies in (the cover space of) \mathcal{L}_A , and intersect with the plane $\mathcal{P}_{[u_{ab}; u_a, v_a]}(\eta')$ with $\eta' = \eta_0 + \delta\eta$ at another pair of points $[u_{ab}(\eta'), v_{ab}^{(\alpha)}(\eta'); u_a(\eta'), v_a(\eta')]$ and $[u_{ab}(\eta'), \tilde{v}_{ab}^{(\tilde{\alpha})}(\eta'); u_a(\eta'), v_a(\eta')]$. The extended integration contour is denoted by $\mathfrak{C}(\eta')$. We define

$$\delta\mathfrak{C}(\eta_0) = \mathfrak{C}(\eta') \setminus \mathfrak{C}(\eta_0) = c \cup \tilde{c} \quad (4.19)$$

which includes 2 piece of curve extensions $c = [u_{ab}(\eta), v_{ab}^{(\alpha)}(\eta); u_a(\eta), v_a(\eta)]$ and $\tilde{c}^{-1} = [u_{ab}(\eta), \tilde{v}_{ab}^{(\tilde{\alpha})}(\eta); u_a(\eta), v_a(\eta)]$. See Fig. 7.

As a result, δI can be expressed by the line integral along the contour extension $\delta\mathfrak{C}$, which is surely included in the FN coordinate chart:

$$\delta_{\eta} I_{\tilde{\alpha}}^{\alpha} = \int_{c \cup \tilde{c}} \left[\sum_{a < b} v_{ab} du_{ab} + \sum_{a=1}^5 v_a du_a \right] \quad (4.20)$$

By definition, c and \tilde{c}^{-1} coincide once they are projected onto the (u_a, v_a) -plane. Therefore

$$\int_{c \cup \tilde{c}} \sum_{a=1}^5 v_a du_a = 0 \quad (4.21)$$

The contribution from $\sum_{a < b} v_{ab} du_{ab}$ can be computed

$$\delta_{\eta} I_{\tilde{\alpha}}^{\alpha} = \int_{c \cup \tilde{c}} \sum_{a < b} v_{ab} du_{ab} = \sum_{a < b} \delta_{\eta} u_{ab} [v_{ab}^{(\alpha)} - \tilde{v}_{ab}^{(\tilde{\alpha})}] + o(\delta\eta^2) \quad (4.22)$$

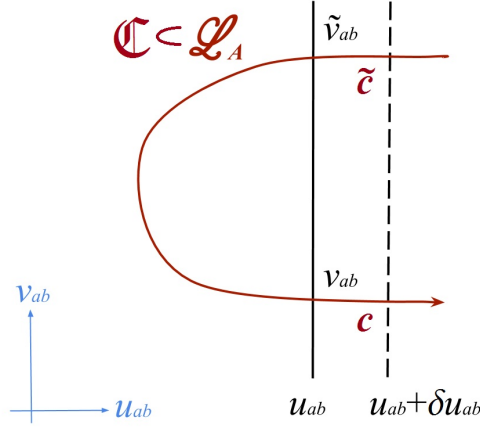


Figure 7. The Lagrangian subvariety \mathcal{L}_A and the plane $\mathcal{P}_{[x_{ab}; x_a, y_a]}$ intersect at 2 different points. The red curve is the integration contour \mathcal{C} lying in \mathcal{L}_A , and connecting the pair of intersection points. The black vertical line represents the plane $\mathcal{P}_{[x_{ab}; x_a, y_a]}$ intersect \mathcal{C} at 2 intersection points. The dashed black vertical line represents the variation $\mathcal{P}_{[x_{ab}; x_a, y_a]}(\eta)$ from $\mathcal{P}_{[x_{ab}; x_a, y_a]}$. $\mathcal{P}_{[x_{ab}; x_a, y_a]}(\eta)$ intersect \mathcal{L}_A at a different pair of points, which is also connected by the extended integration contour $\mathcal{C}(\eta)$. The 2 red segment in between the black line and dashed black line are the curve extensions $\delta\mathcal{C} = c \cup \tilde{c}$. In this figure we suppress the coordinates x_a, y_a .

The variation $\delta\eta$ change the geometry of the 4-simplex but doesn't change the constant curvature Λ . By applying the relation between the FN coordinate and the geometrical quantities Eq.(4.14), we have

$$\delta_\eta I_{\tilde{\alpha}}^\alpha = \left(\frac{\Lambda t}{12\pi i} \right) \sum_{a < b} \delta_\eta \mathbf{a}_{ab} \Theta_{ab} + \left(\frac{\Lambda t}{6} \right) \sum_{a < b} \Delta N_{ab} \delta_\eta \mathbf{a}_{ab}. \quad (4.23)$$

Recall that η parametrizes the geometrical variation of the 4-simplex, the above $\delta_\eta I_{\tilde{\alpha}}^\alpha$ can be integrated and gives

$$I_{\tilde{\alpha}}^\alpha = \left(\frac{\Lambda t}{12\pi i} \right) \left[\sum_{a < b} \mathbf{a}_{ab} \Theta_{ab} - \Lambda \text{Vol}_4^\Lambda + C_{\tilde{\alpha}}^\alpha \right] + \left(\frac{\Lambda t}{6} \right) \sum_{a < b} \Delta N_{ab} \mathbf{a}_{ab} \quad (4.24)$$

The first term contains the Regge action $\sum_{a < b} \mathbf{a}_{ab} \Theta_{ab} - \Lambda \text{Vol}_4^\Lambda$ for a (Lorentzian) constant curvature 4-simplex, up to an integration constant $C_{\tilde{\alpha}}^\alpha$ independent of geometry. Vol_4^Λ denotes the 4-volume of the constant curvature 4-simplex. The Regge action is a function of the Lorentzian geometries of a 4-simplex with constant curvature Λ ²². The derivation of the above result uses the Schläfli identity for a constant curvature 4-simplex:

$$\sum_{a < b} \mathbf{a}_{ab} \delta_\eta \Theta_{ab} = \Lambda \delta_\eta \text{Vol}_4^\Lambda. \quad (4.25)$$

The proof of Schläfli identity can be found in e.g. [43, 79]. See also [80] for the symplectic and semiclassical perspectives of the Schläfli identity. In the expression of $I_{\tilde{\alpha}}^\alpha$, the dependence of branches $\alpha, \tilde{\alpha}$ possibly comes from the integration constant $C_{\tilde{\alpha}}^\alpha$, as well as the term $\left(\frac{\Lambda t}{6} \right) \sum_{a < b} \Delta N_{ab} \mathbf{a}_{ab}$ which only takes discrete values.

Here We have shown the following result: Given a set of boundary data $[x_{ab}; x_a, y_a]$ which is extendible to a flat connection on $S^3 \setminus \Gamma_5$, and geometrically corresponds to the boundary of a convex constant curvature 4-simplex, there exists precisely 2 different $\text{SL}(2, \mathbb{C})$ flat connection A and \tilde{A} on $S^3 \setminus \Gamma_5$ whose boundary values consistent with $[x_{ab}; x_a, y_a]$. We choose arbitrarily 2 branches $\alpha, \tilde{\alpha}$ for lifting A and \tilde{A} to the cover space.

²²Regge action is usually expressed as a function of all the edge-lengths of the 4-simplex.

The holomorphic 3d block in Chern-Simons theory of the lifted flat connection (A, α) with the reference $(\tilde{A}, \tilde{\alpha})$ produces Regge action as the lift-independent term in the leading order \hbar -expansion

$$Z_{CS}^{(\alpha)}(M_3|u) = \exp \left[\frac{i}{\hbar} \left(\frac{\Lambda t}{12\pi i} \right) \left(\sum_{a<b} \mathbf{a}_{ab} \Theta_{ab} - \Lambda \text{Vol}_4^\Lambda \right) + \frac{i}{\hbar} \left(\frac{\Lambda t}{12\pi i} \right) C_{\tilde{\alpha}}^\alpha + \frac{i}{\hbar} \left(\frac{\Lambda t}{6} \right) \sum_{a<b} \Delta N_{ab} \mathbf{a}_{ab} + \dots \right]. \quad (4.26)$$

As the leading and lift-independent contribution in Eq.(4.26), Regge action is a simplicial discretization of Einstein-Hilbert action in General Relativity [43–46]. The above asymptotic expansion suggests that the holomorphic 3d block in Chern-Simons theory on $S^3 \setminus \Gamma_5$ selected by the boundary condition actually gives a wave function of 4-dimensional simplicial gravity with a cosmological constant term.

The leading asymptotic behavior of $Z_{CS}^{(\alpha)}(M_3|u)$ is not an oscillating phase. The oscillating asymptotics may be preferred here because of the relation with quantum simplicial gravity. It can be achieved by considering the full $\text{SL}(2, \mathbb{C})$ Chern-Simons theory Eq.(4.1) including both holomorphic and anti-holomorphic parts.

$$\begin{aligned} & Z_{CS}^{(\alpha)}(M_3|u) Z_{CS}^{(\bar{\alpha})}(M_3|\bar{u}) \\ &= \exp \left[\frac{i}{\hbar} 2\text{Re} \left(\frac{\Lambda t}{12\pi i} \right) \left(\sum_{a<b} \mathbf{a}_{ab} \Theta_{ab} - \Lambda \text{Vol}_4^\Lambda \right) + \frac{i}{\hbar} 2\text{Re} \left(\frac{\Lambda t}{12\pi i} C_{\tilde{\alpha}}^\alpha \right) + \frac{i}{\hbar} 2\text{Re} \left(\frac{\Lambda t}{6} \right) \sum_{a<b} \Delta N_{ab} \mathbf{a}_{ab} + \dots \right] \end{aligned} \quad (4.27)$$

The anti-holomorphic block $Z_{CS}^{(\bar{\alpha})}(M_3|\bar{u})$ is defined by complex conjugate of $(u, v^{(\alpha)})$ with the reference being complex conjugate to $(u, \tilde{v}^{(\bar{\alpha})})$.

The 3d block $Z_{CS}^{(\alpha)}(M_3|u) Z_{CS}^{(\bar{\alpha})}(M_3|\bar{u})$ of full $\text{SL}(2, \mathbb{C})$ Chern-Simons theory gives an oscillating phase with the simplicial Einstein-Hilbert action, as the leading order in \hbar , while the $\log \hbar$ and $\sum_{n=0}^\infty S_n^{(\alpha)}(u) \hbar^n$ in “...” provide the quantum correction to the classical gravity. Eq.(4.26) suggests the following identification of the gravitational constant G_N in 4 dimensions:

$$G_N = \left| \frac{3}{2\text{Im}(t)\Lambda} \right|. \quad (4.28)$$

The term $\frac{i}{\hbar} 2\text{Re} \left(\frac{\Lambda t}{6} \right) \sum_{a<b} \Delta N_{ab} \mathbf{a}_{ab}$ depends on the 4-simplex geometry and depends on the lift by $\Delta N_{ab} \in \mathbb{Z}$. For removing the ambiguity, this term doesn't affect the asymptotic behavior when

$$2\text{Re} \left(\frac{\Lambda t}{6} \right) \sum_{a<b} \Delta N_{ab} \mathbf{a}_{ab} \in 2\pi \hbar \mathbb{Z} \quad (4.29)$$

Interestingly the above condition is satisfied trivially when t is purely imaginary. However for generic complex t , Eq.(4.29) imposes the quantization condition to the triangle areas \mathbf{a}_{ab} , i.e. the triangle area \mathbf{a}_{ab} can only take discrete values.

The integration constant $C_{\tilde{\alpha}}^\alpha$ in Eq.(4.27) is independent of geometry, so it is unimportant as far as the variational principle for geometry is concerned.

An example is presented in the next section, in which the boundary condition is imposed by a Wilson graph operator associated to Γ_5 . In the example, the Chern-Simons coupling t is complex with $k = \text{Re}(t)$ being a nonzero integer. The quantization condition Eq.(4.29) turns out to be indeed satisfied. Moreover the integration constant $C_{\tilde{\alpha}}^\alpha$ vanishes in the example. Therefore the resulting asymptotics is completely independent of the choice of lifts.

The Wilson graph operator used in the next section is well-related to the framework of covariant Loop Quantum Gravity (LQG), as it is explained in Section 6. The boundary data preferred by the kinematical framework of LQG always leads Eq.(4.29) to be true, because of the quantized area spectrum in LQG.

5 Wilson Graph Operator and Boundary Conditions

The analysis in the previous sections studies Chern-Simons theory on a Γ_5 -graph complement 3-manifold M_3 with certain boundary condition (Section 2.2) imposed on the boundary $\partial M_3 = \Sigma_6$. However it is interesting to see that the boundary condition can be imposed by a quantum state, or equivalently an operator in Chern-Simons theory. In this section, we restrict to the case that $\text{Re}(t) = k \in \mathbb{Z}$ and the scaling parameter $\hbar^{-1} \in \mathbb{Z}$.

The state/operator imposing the boundary condition is motivated by the following idea: By the general feature of Topological Quantum Field Theory (see e.g. [8]), a Hilbert space $\mathcal{H}(\Sigma_6)$ is associated to the closed 2-surface Σ_6 . The partition function $Z_{CS}(M_3)$ in Eq.(4.2) is understood as a quantum state in $\mathcal{H}(\Sigma_6)$. We consider the following inner product

$$\mathcal{A}[u_{ab}; u_a, v_a] := \left\langle \Psi_{[u_{ab}; u_a, v_a]}^{\Gamma_5} \middle| Z_{CS}(M_3) \right\rangle_{\mathcal{H}(\Sigma_6)} \quad (5.1)$$

The state $\Psi_{[u_{ab}; u_a, v_a]}^{\Gamma_5} \in \mathcal{H}(\Sigma_6)$ imposes the boundary condition with the boundary data $[u_{ab}; u_a, v_a]$.

The state $\Psi_{[u_{ab}; u_a, v_a]}^{\Gamma_5}$ can be defined by using a Wilson graph operator in the following way: Let's consider the 3-manifold $N(\Gamma_5)$ being the tubular neighborhood of the Γ_5 -graph embedded in S^3 , i.e. $N(\Gamma_5)$ is a solid genus-6 torus, the boundary of $N(\Gamma_5)$ is $\bar{\Sigma}_6$ with opposite induce orientation comparing to $\Sigma_6 = \partial S^3 \setminus \Gamma_5$. We consider $\text{SL}(2, \mathbb{C})$ Chern-Simons theory on $N(\Gamma_5)$ and define a knotted graph operator $\Gamma_5[j_{ab}, \xi_{ab}]$ located inside $N(\Gamma_5)$ [20], such that the state $\Psi_{[u_{ab}; u_a, v_a]}^{\Gamma_5}$ can be written as a functional integration

$$\Psi_{[u_{ab}; u_a, v_a]}^{\Gamma_5}(A_1, \bar{A}_1) := \int_{A_1, \bar{A}_1} \mathcal{D}A \mathcal{D}\bar{A} e^{\frac{i}{\hbar} CS[N(\Gamma_5)|A, \bar{A}]} \Gamma_5[j_{ab}, \xi_{ab}]. \quad (5.2)$$

Here $CS[N(\Gamma_5)|A, \bar{A}]$ replaces M_3 in Eq.(4.1) by $N(\Gamma_5)$, while in the boundary terms $\partial N(\Gamma_5) = \bar{\Sigma}_6$ has the opposite orientation to ∂M_3 . The relation between the operator label $[j_{ab}, \xi_{ab}]$ and the state label $[u_{ab}; u_a, v_a]$ is clarified in the following discussion. By the above definition of $\Psi_{[u_{ab}; u_a, v_a]}^{\Gamma_5}$, the inner product Eq.(5.1) is a functional integration of Chern-Simons theory on $S^3 = N(\Gamma_5) \cup (S^3 \setminus \Gamma_5)$ by gluing Eq.(5.2) and Eq.(4.2):

$$\mathcal{A}[u_{ab}; u_a, v_a] = \int \mathcal{D}A \mathcal{D}\bar{A} e^{\frac{i}{\hbar} CS[S^3|A, \bar{A}]} \Gamma_5[j_{ab}, \xi_{ab}|A, \bar{A}]. \quad (5.3)$$

The knotted graph operator $\Gamma_5[j_{ab}, \xi_{ab}|A, \bar{A}]$ in the path integral is defined in the following (see also [20]):

- Each edge ℓ_{ab} of Γ_5 -graph is associated with a Wilson line operator of unitary irrep of $\text{SL}(2, \mathbb{C})$ labeled by $(j_{ab}, \gamma_{j_{ab}})$ where $j_{ab} \in \mathbb{Z}_+/2$. Recall that the principle series of (infinite-dimensional) unitary $\text{SL}(2, \mathbb{C})$ irrep is classified by using 2 parameters (j, ρ) with a discrete parameter $j \in \mathbb{Z}/2$ and continuous real parameter $\rho \in \mathbb{R}$ [81]. Here we require that the 2 parameters have a constant ratio $\gamma = \rho_{ab}/j_{ab}$ for all edges ℓ_{ab} ²³. The $\text{SL}(2, \mathbb{C})$ unitary irrep $\mathcal{H}^{j\rho}$ can be decomposed into an infinite tower of $\text{SU}(2)$ irreps $\mathcal{H}^{j\rho} = \oplus_{k \geq j} V_k$, where V_k is the $\text{SU}(2)$ irrep with spin $k \in \mathbb{Z}_+/2$. By the decomposition, the canonical basis in $\mathcal{H}^{j\rho}$ is denoted by $|(j, \rho); k, m\rangle$.
- Given an edge ℓ_{ab} in Γ_5 -graph, each of the 2 end points is associated with an $\text{SU}(2)$ coherent state $|j_{ab}, \xi_{ab}\rangle$ or $|j_{ba}, \xi_{ba}\rangle$. $|j, \xi\rangle$ is a vector in the unitary irrep V_j of $\text{SU}(2)$ labeled by the spin j , defined by an $\text{SU}(2)$ action on the highest weight vector [82]:

$$|j, \xi\rangle := g(\xi)|j, j\rangle, \quad g(\xi) \equiv \begin{pmatrix} \xi^1 & -\bar{\xi}^2 \\ \xi^2 & \bar{\xi}^1 \end{pmatrix}. \quad (5.4)$$

Here ξ is a normalized 2-spinor by Hermitian inner product $\bar{\xi}^1 \xi^1 + \bar{\xi}^2 \xi^2 = 1$. $g(\xi) \in \text{SU}(2)$ rotates the 3-vector $\hat{z} = (0, 0, 1)$ to the unit vector $\hat{n}(\xi) = \langle \xi, \vec{\sigma} \xi \rangle$, where $\vec{\sigma}$ are Pauli matrices. Thus the

²³ γ corresponds to the Barbero-Immirzi parameter in LQG

coherent state is essentially labeled by a unit 3-vector \hat{n} . The class of coherent states $|j, \xi\rangle$ forms an over-complete basis in V_j and satisfies the resolution of identity $\mathbf{1}_j = (2j+1) \int_{S^2} d\mu(\xi) |j, \xi\rangle\langle j, \xi|$. The integration domain is the coset $S^2 = \text{SU}(2)/\text{U}(1)$ since $|j, \xi\rangle \mapsto e^{i\phi}|j, \xi\rangle = |j, e^{i\phi}\xi\rangle$ leaves the integrand invariant. The phase convention for ξ needs to be canonically fixed for defining the coherent state basis. We define an injection $Y : V_j \hookrightarrow \mathcal{H}^{j, \gamma j}$ by identifying the $\text{SU}(2)$ irrep \mathcal{H}_j to the lowest subspace in the tower $\mathcal{H}^{j, \gamma j} = \oplus_{k \geq j} \mathcal{H}_k$. We denote $Y|j, \xi\rangle = |(j, \gamma j); j, \xi\rangle$. Therefore each edge ℓ_{ab} is associated with the coherent states $|(j_{ab}, \gamma j_{ab}); j_{ab}, \xi_{ab}\rangle$ and $|(j_{ab}, \gamma j_{ab}); j_{ab}, \xi_{ba}\rangle$.

- The Wilson graph operator $\Gamma_5 [j_{ab}, \xi_{ab} | A, \bar{A}]$ is defined by a product over all edges ℓ_{ab} of inner products in $\mathcal{H}^{j_{ab}, \gamma j_{ab}}$ ²⁴:

$$\Gamma_5 [j_{ab}, \xi_{ab} | A, \bar{A}] := \int_{\text{SL}(2, \mathbb{C})} \prod_{a=1}^5 dg_a \prod_{a < b} \langle (j_{ab}, \gamma j_{ab}); j_{ab}, \xi_{ab} | g_a^{-1} G_{ab} g_b | (j_{ab}, \gamma j_{ab}); j_{ab}, \xi_{ba} \rangle. \quad (5.5)$$

Here the inner products take place in the unitary representation $(j_{ab}, \gamma j_{ab})$ of the $\text{SL}(2, \mathbb{C})$. $G_{ab} = \mathcal{P} \exp \int_{\ell_{ab}} A$ is the holonomy of A along ℓ_{ab} oriented from b to a . $\Gamma_5 [j_{ab}, \xi_{ab} | A, \bar{A}]$ is gauge invariant by the Haar integrals $\int_{\text{SL}(2, \mathbb{C})} \prod_{a=1}^5 dg_a$. Note that the above inner products are not the holomorphic functions on the complex group $\text{SL}(2, \mathbb{C})$, since they come from the unitary representations.

- Because of the invariance of $e^{\frac{i}{\hbar} CS}$ with $k \in \mathbb{Z}$ under large gauge transformation, it is convenient to make a partial gauge fixing at the vertices in the functional integrations Eqs.(5.2) and (5.3), by setting all $g_a = 1$ for $a = 1, \dots, 5$ and dropping all the g_a -integrals in Eq.(5.5). Such a gauge fixing removes the divergences of Wilson graph operator. In the following we still denote the gauge-fixed Wilson graph operator by $\Gamma_5 [j_{ab}, \xi_{ab} | A, \bar{A}]$.

Note that the Γ_5 graph is associated with a framing, which is a vector field defined on the graph projected on a plane as in Fig. 2, with all the vectors pointing toward the reader from the paper.

We can factorize the knotted graph operator $\Gamma_5 [j_{ab}, \xi_{ab} | A, \bar{A}]$ into the ingredients associated to each edge and the ingredients associated to the neighborhood of each vertex of Γ_5 -graph, namely, we rewrite the inner products in $\Gamma_5 [j_{ab}, \xi_{ab} | A, \bar{A}]$ as

$$\int_{\mathbb{CP}^1} dz_{ab} \int_{\mathbb{CP}^1} dz_{ba} \langle (j_{ab}, \gamma j_{ab}); j_{ab}, \xi_{ab} | G_a^{(\ell_{ab})-1} | z_{ab} \rangle \langle z_{ab} | G'_{ab} | z_{ba} \rangle \langle z_{ba} | G_b^{(\ell_{ab})} | (j_{ab}, \gamma j_{ab}); j_{ab}, \xi_{ba} \rangle \quad (5.6)$$

where we have split each edge ℓ_{ab} into 3 pieces and rewritten the holonomy G_{ab} into $G_a^{(\ell_{ab})-1} G'_{ab} G_b^{(\ell_{ab})}$. $\langle z | f \rangle$ denotes the representation of the vector $|f\rangle \in \mathcal{H}^{j, \rho}$ by homogeneous function of 2 complex variables $f(z) \equiv f(z^1, z^2, \bar{z}^1, \bar{z}^2)$. The inner product is of L^2 -type in this representation $\langle f | f' \rangle = \int_{\mathbb{CP}^1} dz \bar{f}(z) f'(z)$ with $dz = \frac{i}{2} (z^1 dz^2 - z^2 dz^1) \wedge (\bar{z}^1 d\bar{z}^2 - \bar{z}^2 d\bar{z}^1)$. For the reviews on the representation of $\text{SL}(2, \mathbb{C})$ in terms of homogeneous functions, see e.g. [81], see also [84] for a brief summary.

The factor $\langle z_{ab} | G'_{ab} | z_{ba} \rangle$ in the above integral can be written as a path integral formula by using the quantization of coadjoint orbit for $\text{SL}(2, \mathbb{C})$. The idea is that the unitary irreducible representations of a Lie group G can be obtained by the geometrical quantization of its coadjoint orbit. The reviews of the topic can be found in [85] (see also [24] for a nice summary). Here we also provide a brief review in Appendix F. The path integral formula of $\langle z_{ab} | G'_{ab} | z_{ba} \rangle$ in $\mathcal{H}^{j_{ab}, \gamma j_{ab}}$ reads

$$\langle z_{ab} | G'_{ab} | z_{ba} \rangle = \int_{z_{ba}}^{z_{ab}} \mathcal{D}g_{ab} \mathcal{D}\bar{g}_{ab} e^{iS'_{ab}[g_{ab}, \bar{g}_{ab}; A, \bar{A}]}, \quad (5.7)$$

where the Lagrangian $S'_{ab}[g_{ab}, \bar{g}_{ab}; A, \bar{A}]$ is given by:

$$S'_{ab}[g_{ab}, \bar{g}_{ab}; A, \bar{A}] = -\frac{1}{2} \int_{\ell'_{ab}} \text{tr} \left[(\nu + \kappa) g_{ab}^{-1} (d + A^T) g_{ab} + (\nu - \kappa) \bar{g}_{ab}^{-1} (d + \bar{A}^T) \bar{g}_{ab} \right] \quad (5.8)$$

²⁴The knotted graph operator relates to the project spin-network function of $\text{SL}(2, \mathbb{C})$ [51, 83].

The path integral is over the maps $g_{ab} : \ell'_{ab} \rightarrow \text{SL}(2, \mathbb{C})$ where ℓ'_{ab} is the middle part of edge ℓ_{ab} defining G'_{ab} . However there is a gauge symmetry of the Lagrangian, i.e. $S'_{ab}[g, \bar{g}; A, \bar{A}]$ is invariant under $g \mapsto gh$ with h in the Cartan subgroup $\mathbb{T}_{\mathbb{C}}$. Therefore the path integral is essentially defined to be over the maps $g_{ab} : \ell'_{ab} \rightarrow \text{SL}(2, \mathbb{C})/\mathbb{T}_{\mathbb{C}}$. The path integral has a first-order Lagrangian. The boundary condition for the path integral is that the “position variables” of g_{ab} at the source and target of ℓ'_{ab} equal to z_{ba} and z_{ab} . The above path integral can be viewed as a quantum particle moving on its “position space” \mathbb{CP}^1 . In the Lagrangian, ν, κ are the 2×2 matrices corresponding to the weight:

$$\nu = -\gamma j_{ab} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \kappa = i j_{ab} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (5.9)$$

Let's focus on the state $\Psi_{[u_{ab}, u_a, v_a]}^{\Gamma_5}$ defined on $N(\Gamma_5)$. We zoom in a tubular neighborhood $N(\ell'_{ab}) \subset N(\Gamma_5)$ of ℓ'_{ab} (ℓ'_{ab} is the middle part of the edge ℓ_{ab} for Eqs.(5.7) and (5.8)). $N(\ell'_{ab})$ has the topology $[0, 1] \times D$ where D is a 2-disk. The Chern-Simons connection in $N(\ell'_{ab})$ can be decomposed into a “time component” A_t along ℓ'_{ab} and “spatial component” A_{\perp} . With the decomposition and after an integration by part, $CS[N(\Gamma_5)|A, \bar{A}]$ with integration domain restricted in $N(\ell'_{ab})$ becomes²⁵

$$CS[N(\ell'_{ab})|A, \bar{A}] = \frac{t}{8\pi} \int_{N(\ell'_{ab})} \text{tr}(A_{\perp} \wedge dA_{\perp}) + \frac{t}{4\pi} \int_{N(\ell'_{ab})} \text{tr}(F_{\perp} \wedge A_t) + \text{c.c.} \quad (5.10)$$

where $F_{\perp} = dA_{\perp} + A_{\perp} \wedge A_{\perp}$ is the curvature. The Chern-Simons theory on $N(\ell'_{ab})$ is coupled with the path integral Eq.(5.7). The coupled action is linear to A_t, \bar{A}_t . Then we integrate out A_t, \bar{A}_t to get 2 delta functions on the space of A_{\perp} , which constrains F_{\perp} and \bar{F}_{\perp} :

$$\begin{aligned} \frac{t}{4\pi\hbar} F_{\perp}^T &= \frac{1}{2} g(\nu + \kappa) g^{-1} \delta^{(2)}(x) dx_1 \wedge dx_2 \\ \frac{\bar{t}}{4\pi\hbar} \bar{F}_{\perp}^T &= \frac{1}{2} \bar{g}(\nu - \kappa) \bar{g}^{-1} \delta^{(2)}(x) dx_1 \wedge dx_2 \end{aligned} \quad (5.11)$$

We have chosen a local coordinate (x_1, x_2) on D so that the Wilson line goes through the origin. $\delta^{(2)}(x)$ is a delta function on D such that $\int_{\ell'_{ab}} f = \int_{N(\ell'_{ab})} \delta^{(2)}(x) dx_1 \wedge dx_2 \wedge f$ for any 1-form f . Since $F_{\perp}, \bar{F}_{\perp}$ is defined up to conjugation, the above constraints restrict the conjugacy class of the meridian holonomies H_{ab} (same for \bar{H}_{ab}), i.e.

$$H_{ab} \sim \begin{pmatrix} q^{j_{ab}} & 0 \\ 0 & q^{-j_{ab}} \end{pmatrix}, \quad \text{where } q = e^{\frac{2\pi i \hbar}{t}(1+i\gamma)}. \quad (5.12)$$

When the parameters t, γ satisfy $\frac{2\pi i \hbar}{t}(1+i\gamma) \in \mathbb{R}$, the eigenvalue $x_{ab} = q^{j_{ab}} \in U(1)$ satisfies the boundary condition in Section 2.2. We apply the constraint to the first term in $CS[N(\ell'_{ab})|A, \bar{A}]$ in Eq.(5.10), by $\text{tr}(A_{\perp} \wedge dA_{\perp}) = \text{tr}(A_{\perp} \wedge F_{\perp})$. We find this term vanishes identically since F_{\perp} is constrained to be proportional to $dx_1 \wedge dx_2$. As a result, the contributions to the wave function $\Psi_{[u_{ab}, u_a, v_a]}^{\Gamma_5}$ from $N(\ell'_{ab})$ for all ℓ'_{ab} give a product of delta functions:

$$\prod_{a < b} \delta_{q^{j_{ab}}}(x_{ab}) \delta_{\bar{q}^{j_{ab}}}(\bar{x}_{ab}). \quad (5.13)$$

The boundary data $x_{ab} = q^{j_{ab}}$ is imposed strongly by the Wilson graph operator.

When we take the semiclassical limit $\hbar \rightarrow 0$, the representation labels j_{ab} has to be sent uniformly to infinity, in order to keep the boundary data $x_{ab} = q^{j_{ab}} = e^{\frac{2\pi i \hbar}{t}(1+i\gamma)j_{ab}}$ invariant, as the analysis in the previous sections. Therefore in the context of Chern-Simons theory coupled with Wilson graph, the right semiclassical limit is a double-scaling limit: $\hbar \rightarrow 0$ and $j_{ab} \rightarrow \infty$ keeping $j_{ab}\hbar$ fixed.

²⁵On the boundary $A_1 = A_{\perp}$ and $A_t = A_2$, the boundary term from integration by part cancels the boundary term $\frac{t}{8\pi} \int_C \text{tr}(A_1 \wedge A_2)$ with $C = \partial N(\ell'_{ab}) \cap \bar{\Sigma}_6$ being a cylinder.

The double-scaling limit for the Chern-Simons theory on S^3 coupled with Γ_5 Wilson graph operator has been studied in the stationary phase method [20]. The Γ_5 Wilson graph operator can be written as in the integral form

$$\Gamma_5 [j_{ab}, \xi_{ab} | A, \bar{A}] := \int_{\mathbb{CP}^1} \prod_{a < b} dZ_{ab} e^{I_{\Gamma_5}} \quad (5.14)$$

where I_{Γ_5} is given by

$$I_{\Gamma_5} = \sum_{a < b} j_{ab} \ln \frac{\langle G_{ab}^\dagger Z_{ab}, \xi_{ba} \rangle^2 \langle \xi_{ab}, Z_{ab} \rangle^2}{\langle G_{ab}^\dagger Z_{ab}, G_{ab}^\dagger Z_{ab} \rangle \langle Z_{ab}, Z_{ab} \rangle} + i\gamma j_{ab} \ln \frac{\langle G_{ab}^\dagger Z_{ab}, G_{ab}^\dagger Z_{ab} \rangle}{\langle Z_{ab}, Z_{ab} \rangle}. \quad (5.15)$$

The stationary phase analysis for I_{Γ_5} coupled with Chern-Simons theory on S^3 in Eq.(5.3) gives the following critical equations:

Parallel Transportation: From $\delta_{Z_{ab}} I_{\Gamma_5} = 0$ and $\text{Re}(I_{\Gamma_5}) = 0$, we have obtained the following parallel transportation relation for the coherent state labels ξ_{ab} :

$$J_{\xi_{ab}} = \frac{\|G_{ab}^\dagger Z_{ab}\|}{\|Z_{ab}\|} e^{-i\theta_{ab}} G_{ab} J_{\xi_{ba}}, \quad \xi_{ab} = \frac{\|Z_{ab}\|}{\|G_{ab}^\dagger Z_{ab}\|} e^{i\theta_{ab}} G_{ab} \xi_{ba}. \quad (5.16)$$

which relates the 2-spinors ξ_{ab} and ξ_{ba} at the 2 different end-points of the edge ℓ_{ab} .

Monodromies: The variation with respect to Chern-Simons connection A, \bar{A} gives the distributional curvature on S^3 ,

$$\varepsilon^{\mu\rho\sigma} F_{\rho\sigma}^i(x) = \frac{8\pi\hbar(1+i\gamma)}{t} \sum_{a < b} j_{ab} \langle G_{sb}^\dagger \sigma_i (G_{sb}^\dagger)^{-1} \xi_{ba}, \xi_{ba} \rangle \delta_{\ell_{ab}}^{(2)\mu}(x). \quad (5.17)$$

where $\delta_{\ell}^{(2)\mu}(x) := \int_0^1 \delta^{(3)}(x - \ell(s)) \frac{d\ell^\mu}{ds} ds$ and σ_i denote Pauli matrices. \bar{F} is obtained by the complex conjugations of the above equations. The parameter s labeling the holonomy G_{sb} means an arbitrary middle point on the edge ℓ_{ab} , which coincides with x when x is on ℓ_{ab} . Thus G_{sb} is the holonomy starting at the vertex b and ending at the middle point s . Integrating F gives the nontrivial holonomies along the small non-contractible cycles c_{ab} transverse to each edge ℓ_{ab} :

$$H_{ab}(s) = \exp \left[\frac{4\pi\hbar(1+i\gamma)}{t} j_{ab} \langle G_{sb}^\dagger \sigma_j (G_{sb}^\dagger)^{-1} \xi_{ba}, \xi_{ba} \rangle \frac{i\sigma_j}{2} \right], \quad (a < b). \quad (5.18)$$

The curvature is only supported distributionally on the graph Γ_5 , while $F = \bar{F} = 0$ on the graph complement $S^3 \setminus \Gamma_5$.

The conjugacy class of $H_{ab}(s)$ is consistent with the delta function Eq.(5.13). Let's consider a 2-sphere with radius s enclosing the vertex a , and denote $H_{ab_l}(s) \equiv H_l(s)$ ($l = 1, \dots, 4$). By integrating the curvature F on the 2-sphere using non-abelian stokes theorem, we obtain that

$$g_4(s) H_4(s) g_4(s)^{-1} g_3(s) H_3(s) g_3(s)^{-1} g_2(s) H_2(s) g_2(s)^{-1} g_1(s) H_1(s) g_1(s)^{-1} = 1 \quad (5.19)$$

where $g_l \in \text{SL}(2, \mathbb{C})$ is the holonomy connecting the base point of each $H_l(s)$ to an arbitrary common base point on the sphere. Because of $F = \bar{F} = 0$ outside the graph²⁶,

$$g_l(s)^{-1} g_{l-1}(s) = G_{as_l}^{-1} G_{as_{l-1}}. \quad (5.20)$$

²⁶The holonomy G has been lifted to the boundary Σ_6 of graph complement $S^3 \setminus \Gamma_5$. The lift is taken in the direction defined by the framing vector field on Γ_5 .

On the other hand, using Eq.(5.16) one can show that each $H_l(s)$ is conjugated to $SU(2)$ by $G_{as_l}^{-1}$, when the parameters t, γ satisfy $\frac{2\pi\hbar}{t}(1+i\gamma) \in \mathbb{R}$.

$$G_{as}H_{ab}(s)G_{as}^{-1} = H_{ab}(a) = \exp\left[\frac{4\pi\hbar(1+i\gamma)}{t}j_{ab}\hat{n}_{ab}\frac{i\sigma_j}{2}\right] \quad (5.21)$$

where $\hat{n}_{ab} = \langle \xi_{ab}, \vec{\sigma}\xi_{ab} \rangle$ is a unit vector in \mathbb{R}^3 . Then Eq.(5.19) reduces to a product of four $SU(2)$ matrices

$$\overleftarrow{\prod} \exp\left[\frac{4\pi\hbar(1+i\gamma)}{t}j_{ab}\hat{n}_{ab}\frac{i\sigma_j}{2}\right] = 1 \quad (5.22)$$

It shows that after removing the intersection points (and their neighborhood) between Γ_5 and the sphere enclosing a , the connection on the resulting 4-holed sphere is essentially an $SU(2)$ flat connection. Notices that these 4-holed spheres for all vertices are essentially $\{\mathcal{S}_{a=1,\dots,5}\} = \Sigma_6 \setminus \{c_{ab}\}_{a<b}$. Thus the boundary condition in Section 2.2 is derived from the Γ_5 Wilson graph operator via double-scaling limit.

The boundary data label $[u_{ab}; u_a, v_a]$ of the state $\Psi_{[u_{ab}; u_a, v_a]}^{\Gamma_5}$ is given by (1) $u_{ab} = \frac{2\pi\hbar}{t}(1+i\gamma)j_{ab}$ imposed strongly as a delta function, (2) u_a, v_a parametrize the shape of tetrahedron reconstructed in Section 2.3 from Eq.(5.22). The boundary data u_a, v_a is imposed only weakly in the semiclassical limit, being consistent with Heisenberg's uncertainty principle. In this sense, the state $\Psi_{[u_{ab}; u_a, v_a]}^{\Gamma_5}$ can be viewed as the ‘‘semiclassical’’ state peaked at the phase space point (u_a, v_a) product with the delta functions $\delta_{x_{ab}}$.

The solution of critical equations is an $SL(2, \mathbb{C})$ flat connection on the graph complement 3-manifold $S^3 \setminus \Gamma_5$. There are two solutions A, \bar{A} being a parity pair by Theorem 2.5. Because the Wilson graph operator imposes properly the boundary condition, the $SL(2, \mathbb{C})$ flat connection on $S^3 \setminus \Gamma_5$ satisfying the critical equations corresponds to a unique convex constant curvature 4-simplex in Lorentzian signature. The representation labels j_{ab} relate to the triangle area a_{ab} by

$$\frac{2\pi i\hbar}{t}\left(\frac{1}{\gamma} + i\right)\gamma j_{ab} = -iv\frac{\Lambda}{6}a_{ab} + i\pi s_{ab} \mod 2\pi i\mathbb{Z}. \quad (5.23)$$

Let's come back to the lift-dependent term in Eq.(4.27), and the quantization condition Eq.(4.29). From the above relation between a_{ab} and j_{ab} , it is straightforward to check that

$$2\text{Re}\left(\frac{\Lambda t}{6}\right)\sum_{a<b}\Delta N_{ab}a_{ab} = -v\sum_{a<b}N_{ab}[2\pi\hbar(2j_{ab}) - 2\pi k s_{ab} + 4\pi k\mathbb{Z}] \in 2\pi\hbar\mathbb{Z} \quad (5.24)$$

indeed satisfies the quantization condition Eq.(4.29).

So far we have shown the boundary data imposed by the Wilson graph operator $\Gamma_5[j_{ab}, \xi_{ab}|A, \bar{A}]$ satisfy the boundary condition proposed in Section 2.2, and satisfy the quantization condition Eq.(4.29). By inserting the solution of critical equation (i.e. a flat connection corresponding to a geometrical 4-simplex) back into the total action $I_{\Gamma_5} + \frac{i}{\hbar}CS$, we obtain the leading asymptotic behavior of $\mathcal{A}[u_{ab}; u_a, v_a]$ in the semiclassical limit. It is not surprising that the asymptotics of $\mathcal{A}[u_{ab}; u_a, v_a]$ is the same as the 3d block in Eq.(4.27), which gives the 4-dimensional Regge action of the constant curvature 4-simplex with a cosmological constant term. In this way, we see that the asymptotic behavior of Chern-Simons 3d block basically determines the asymptotics of $\mathcal{A}[u_{ab}; u_a, v_a]$, while the Wilson graph operator plays of imposing the right boundary condition.

Moreover one can also show that in this case the integration constant $\frac{i}{\hbar}2\text{Re}\left(\frac{\Lambda t}{12\pi i}C_{\bar{\alpha}}^{\alpha}\right)$ vanishes mod $2\pi i\mathbb{Z}$, by a crosscheck between perturbative and nonperturbative computations of Chern-Simons functional [20].

In [20], the following result is shown: Under the double-scaling limit $\hbar \rightarrow 0$, $j_{ab} \rightarrow \infty$ with $j_{ab}\hbar$ fixed, the Chern-Simons expectation value $\mathcal{A}[u_{ab}; u_a, v_a]$ of Γ_5 graph operator in Eq.(5.3) has the following asymptotic behavior

$$\mathcal{A}[u_{ab}; u_a, v_a] = e^{\frac{i}{\ell_P^2}S_{\text{Regge}}^{\Lambda}+\dots} + e^{-\frac{i}{\ell_P^2}S_{\text{Regge}}^{\Lambda}+\dots} \quad (5.25)$$

up to an overall phase factor. The two exponentials comes from the two solutions A and \tilde{A} respectively. \dots stands for the terms of $o(\log \hbar)$ and $\sum_{n=0}^{\infty} S_n \hbar^n$. The constant curvature Regge action of simplicial gravity $S_{\text{Regge}}^{\Lambda}$ reads

$$S_{\text{Regge}}^{\Lambda} = \sum_{a < b} \mathbf{a}_{ab} \Theta_{ab} - \Lambda \text{Vol}_4^{\Lambda} \quad (5.26)$$

and $\ell_P^{-2} = \text{Re} \left(\frac{\Lambda t}{12\pi i \hbar} \right)$. We have assumed here the Chern-Simons couplings $t = k + is$ and $\bar{t} = k - is$ satisfy $k \in \mathbb{Z}$ and $s \in \mathbb{R}$. \hbar^{-1} is an integer being the scaling parameter for t, \bar{t} .

In the semiclassical limit, $\mathcal{A}[u_{ab}; u_a, v_a]$ has the same asymptotic behavior as the sum of a pair of Chern-Simons 3d blocks (up to an overall phase):

$$\mathcal{A}[u_{ab}; u_a, v_a] \sim Z_{CS}^{(\alpha)}(u) Z_{CS}^{(\bar{\alpha})}(\bar{u}) + Z_{CS}^{(\tilde{\alpha})}(u) Z_{CS}^{(\bar{\tilde{\alpha}})}(\bar{u}) \quad (5.27)$$

$Z_{CS}^{\alpha}(u), Z_{CS}^{\tilde{\alpha}}(u)$ corresponds to the pair of flat connection $A, \tilde{A} \in \mathcal{M}_{\text{flat}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C}))$ with 2 arbitrary lifts $\alpha, \tilde{\alpha}$ respectively. Note that the analysis in Section 4.2 have only a single exponential because we compute the phase difference (or ratio) between two 3d blocks $Z_{CS}^{(\alpha)}(u) Z_{CS}^{(\bar{\alpha})}(\bar{u})$ and $Z_{CS}^{(\tilde{\alpha})}(u) Z_{CS}^{(\bar{\tilde{\alpha}})}(\bar{u})$.

In addition, it is interesting that the cosmological constant term in Eq.(5.25) comes from the evaluation of Chern-Simons functional on S^3 at the connection A satisfying critical equations. But the connection is now viewed as a distributional connection on S^3 (with a distributional curvature supported on the graph) instead of being a flat connection on $S^3 \setminus \Gamma_5$. The following difference between the evaluations at A, \tilde{A} gives the constant curvature 4-volume of the 4-simplex:

$$CS[S^3|A, \bar{A}] - CS[S^3|\tilde{A}, \bar{\tilde{A}}] = -\frac{2\Lambda}{\ell_P^2} \text{Vol}_4^{\Lambda} + 2\pi i \mathbb{Z}. \quad (5.28)$$

6 Relation with Loop Quantum Gravity

If we take the asymptotic “decoupling limit” by turning off the Chern-Simons coupling in $\mathcal{A}[u_{ab}; u_a, v_a]$ via $t, \bar{t} \rightarrow \infty$ keeping j_{ab} fixed, the path integral Eq.(5.3) is localized at the solution of Chern-Simons equation of motion $F = \bar{F} = 0$ on S^3 , which gives trivial connection on S^3 . $\Gamma_5[j_{ab}, \xi_{ab}|A, \bar{A}]$ evaluated at trivial connection gives the Engle-Pereira-Rovelli-Livine (EPRL) spinfoam 4-simplex amplitude $\mathcal{A}_{\text{EPRL}}[j_{ab}, \xi_{ab}]$ in LQG. Such a relation is actually the original motivation for the definition of Γ_5 Wilson graph operator.

The relations among Chern-Simons theory, 4-dimensional LQG and 4-dimensional simplicial gravity can be summarized in the following diagram:

$$\begin{array}{ccc} \mathcal{A}[u_{ab}; u_a, v_a] & \xrightarrow{\hbar \rightarrow 0, j \rightarrow \infty, j\hbar \text{ fixed}} & e^{\frac{i}{\ell_P^2} S_{\text{Regge}}^{\Lambda}} + e^{-\frac{i}{\ell_P^2} S_{\text{Regge}}^{\Lambda}} \\ \downarrow t \rightarrow \infty & & \downarrow \Lambda \rightarrow 0 \\ \mathcal{A}_{\text{EPRL}}[j_{ab}, \xi_{ab}] & \xrightarrow{j \rightarrow \infty} & e^{\frac{i}{\ell_P^2} S_{\text{Regge}}} + e^{-\frac{i}{\ell_P^2} S_{\text{Regge}}} \end{array} \quad (6.1)$$

where $\mathcal{A}[u_{ab}; u_a, v_a]$ defined in Eq.(5.3) is the $\text{SL}(2, \mathbb{C})$ Chern-Simons evaluation of Γ_5 Wilson graph operator. The relation downstairs is proved in [55, 56], namely the large- j asymptotics of EPRL spinfoam amplitude reproduces the flat simplicial geometry and the Regge action without cosmological constant Λ . The above relations suggest that the Chern-Simons expectation value $\mathcal{A}[u_{ab}; u_a, v_a]$ can be viewed as a deformation of EPRL spinfoam amplitude, which includes cosmological constant into the framework of LQG.

4-dimensional spinfoam amplitude in LQG, which defines the quantum 4d geometry, describes the quantum transition between boundary states for quantum 3d geometries. The boundary states of 4-dimensional spinfoam amplitude are $\text{SU}(2)$ spin-network states from the kinematical framework of LQG (see [48, 49]),

which describe the quantum 3d geometries. A spin-network states is a triple $(\Gamma, \vec{j}, \vec{i})$. Γ stands for a graph containing oriented edges and vertices. $\vec{j} = \{j_\ell\}_{\ell \in E(\Gamma)}$ is a map from the set of edges $E(\Gamma)$ to the space of unitary irreps of $SU(2)$ labeled by j_ℓ . $\vec{i} = \{i_v\}_{v \in V(\Gamma)}$ is a map from the set of vertices $V(\Gamma)$ to the invariant tensors (intertwiners) $v \mapsto i_v \in \text{Inv}_{SU(2)}(V_{j_1} \otimes \cdots \otimes V_{j_n})$, where j_1, \dots, j_n are the spin labels on the edges adjacent at v . The spin-network state is a basis of LQG Hilbert space and diagonalize the geometrical operators e.g. quantum area and volume operators. The discrete spectrum of area operator relates to the spins j_ℓ (and is linear in j_ℓ when $j_\ell \gg 1$), and the discrete spectrum of volume operator relates to both spins j_ℓ and invariant tensors i_v [47, 88]. Moreover the invariant tensor i_v parametrizes the space of quantum (flat) polyhedra with face areas being proportional to the adjacent j_ℓ [87, 89].

The spin-network data $(\Gamma, \vec{j}, \vec{i})$ is well adapted to the framework in the present paper, by identifying $(\Gamma, \vec{j}, \vec{i})$ to the boundary data of flat connections. In the definition of Γ_5 Wilson graph operator and its Chern-Simons evaluation $\mathcal{A}[u_{ab}; u_a, v_a]$ with $k \in \mathbb{Z}$, the spin-network graph identified to be Γ_5 , which is 4-valent since we are looking at the simplicial geometries. The spins j_ℓ is mapped by Y to an $SL(2, \mathbb{C})$ principle series irrep $(j_\ell, \gamma j_\ell)$ for each edge, where γ is Barbero-Immirzi parameter in LQG. At each vertex, we employ the $SU(2)$ coherent state basis and consider i_v to be a coherent intertwiner, which is mapped by Y to an $SL(2, \mathbb{C})$ invariant tensor in the Wilson graph operator.

Given a graph, e.g. Γ_5 in our context, and its tubular neighborhood $N(\Gamma_5)$, let's consider the quantization of $SU(2)$ flat connections on the closed 2-surface $\Sigma_6 = \partial N(\Gamma_5)$. By specifying the meridian closed curved c_{ab} as we have done in Section 2.2, at least locally in $\mathcal{M}_{\text{flat}}(\Sigma_6, SU(2))$, we quantize a set of symplectic coordinates $x_{ab} = e^{u_{ab}}, y_{ab} = e^{-\frac{\pi}{k} v_{ab}} \in U(1)$ with $\{u_{ab}, v_{ab}\} = 1$, as well as quantize $\mathcal{M}_{\text{flat}}(S_a, SU(2))$ of 4-holed sphere with fixed conjugacy class x_{ab} at each hole. The quantization of x_{ab}, y_{ab} is a quantization of $S^1 \times S^1$. The prequantum line bundle over $S^1 \times S^1$ has a curvature $\omega = -\frac{k}{\pi} d \ln x_{ab} \wedge d \ln y_{ab}$. Weil's integrality criterion then implies that $k \in \mathbb{Z}$. We choose the polarization such that the wave function is written as $f(u_{ab})$, satisfying both periodicity and Weyl invariant $f(u_{ab}) = f(-u_{ab}) = f(u_{ab} + 2\pi i)$. The periodicities in both u_{ab} and v_{ab} directions implies that u_{ab} can only take $k + 1$ discrete values $u_{ab} = 0, \frac{i\pi}{k}, \frac{2i\pi}{k}, \dots, i\pi$, i.e.

$$x_{ab} = e^{\frac{2\pi i}{k} j_{ab}}, \quad j_{ab} = 0, \frac{1}{2}, \dots, \frac{k}{2} \quad (6.2)$$

The quantization of $\mathcal{M}_{\text{flat}}(S_a, SU(2))$ with fixed conjugacy classes x_{ab} results in the Hilbert space $\mathcal{H}(S_a)$ spanned by Wess-Zumino-Witten (WZW) conformal blocks $\mathcal{F}(i_a)$ of level $k \in \mathbb{Z}$ on a 4-holed sphere [65]. Each conformal block $\mathcal{F}(i_a)$ associates with an 4-valent $SU(2)$ intertwiner i_a with the above spins j_{ab} . But only a restricted subclass of $SU(2)$ intertwiners is allowed in $\mathcal{H}(S_a)$. The dimension of $\mathcal{H}(S_a)$ is given by the famous Verlinde formula [90]. As a result, we obtain the Hilbert space from the quantization of $\mathcal{M}_{\text{flat}}(\Sigma_6, SU(2))$, which is spanned by the following basis

$$\psi_{(\Gamma_5, \vec{j}, \vec{i})} = \prod_{a < b} \delta_{\exp[\frac{2\pi i}{k} j_{ab}]} \prod_{a=1}^5 \mathcal{F}(i_a) \quad (6.3)$$

where $\delta_{\exp[\frac{2\pi i}{k} j_{ab}]}$ is a Kronecker delta by the discreteness of j_{ab} or x_{ab} . The above discussion can be trivially generalized to arbitrary graphs Γ . Now we see that the quantization of $SU(2)$ flat connections on $\Sigma_g = \partial N(\Gamma)$ for any graph Γ naturally gives the spin-network data $(\Gamma, \vec{j}, \vec{i})$ with $j \leq k/2$ and a restricted subclass of intertwiners. The restricted class of spin-network data is likely to be the right subclass in LQG with a cosmological constant implemented.

By the analysis in Section 2.3, the $SU(2)$ flat connections on a 4-holed sphere with fixed conjugacy classes x_{ab} correspond to the constant curvature tetrahedra geometries with fixed face areas. Therefore the Hilbert space $\mathcal{H}(S_a)$ of conformal blocks is the space of “quantum constant curvature tetrahedra” with “quantum area” proportional to j_{ab} . We may consider a (overcomplete) coherent state basis ψ_{x_a, y_a}^k peaked at the phase space point with conjugate coordinates x_a, y_a . By the coherent state labels, the above spin-network data $(\Gamma, \vec{j}, \vec{i})$ when $\Gamma = \Gamma_5$ maps to $(x_{ab}; x_a, y_a)$ under the restriction of spins and intertwiners, where $(x_{ab}; x_a, y_a)$ is precisely the boundary data imposed for $SL(2, \mathbb{C})$ flat connections on $\Sigma_6 = \partial S^3 \setminus \Gamma_5$.

In order to be the boundary data of $\text{SL}(2, \mathbb{C})$ Chern-Simons theory, we make the following identification:

$$x_{ab} = e^{\frac{2\pi i}{k} j_{ab}} = e^{\frac{2\pi i}{t} (1+i\gamma) j_{ab}} \quad (6.4)$$

by Eq.(5.13) (setting $\hbar = 1$). Here $k \in \mathbb{Z}$ has been identified with $\text{Re}(t)$, and both γ and $\frac{1}{t} (1 + i\gamma)$ have been assumed to be real numbers, so that $\gamma = s/k$. Given that $(x_{ab}; x_a, y_a)$ comes from a spin-network data, the boundary condition in Section 2.2 is satisfied by $(x_{ab}; x_a, y_a)$, and the quantization condition Eq.(4.29) is also satisfied by the same argument in Section 5. It is interesting to notice that when t is purely imaginary ($k = 0$ or $\gamma \rightarrow \infty$), the spectrum of x_{ab} is not discrete anymore, while the quantization condition Eq.(4.29) is satisfied trivially. This possibility is beyond the regime of spin-network data, but still well-controlled by Chern-Simons 3d block discussed in Section 4.2²⁷.

The above discuss provides us a map from spin-network data to the boundary data $(x_{ab}; x_a, y_a)$ of $\text{SL}(2, \mathbb{C})$ Chern-Simons theory satisfying the boundary condition in Section 2.2. When there exists an $\text{SL}(2, \mathbb{C})$ flat connection A on $S^3 \setminus \Gamma_5$ whose boundary value is consistent with the boundary data $(x_{ab}; x_a, y_a)$, we may use $(x_{ab}; x_a, y_a)$ to construct Chern-Simons 3d block $Z_{CS}^{(a)}(u)Z_{CS}^{(\bar{a})}(\bar{u})$. Chern-Simons 3d block $Z_{CS}^{(a)}(u)Z_{CS}^{(\bar{a})}(\bar{u})$ studied in Section 4.2 may also play an interesting role in LQG in addition to the spinfoam amplitude $\mathcal{A}_{\text{EPRL}}[j_{ab}, \xi_{ab}]$ or $\mathcal{A}[u_{ab}; u_a, v_a]$. As we have seen in Sections 5, the Regge-action asymptotic behavior of $\mathcal{A}[u_{ab}; u_a, v_a]$ crucially depends on the peakedness of Chern-Simons state created by the Wilson graph operator. However different Wilson graph operators may produce the same peakedness of boundary data, thus leads to the same asymptotics of $\mathcal{A}[u_{ab}; u_a, v_a]$. The close relationship with EPRL 4-simplex amplitude motivates us to study the particular type of Wilson graph operators $\Gamma_5[j_{ab}, \xi_{ab}|A, \tilde{A}]$. But in principle other types of Wilson graph operators are possible to work equally well, when they produce the same peakedness. However independent of the choice of Wilson graphs, the essential role played behind the Regge-action asymptotics of $\mathcal{A}[u_{ab}; u_a, v_a]$ is Chern-Simons 3d block $Z_{CS}^{(a)}(u)Z_{CS}^{(\bar{a})}(\bar{u})$ on $S^3 \setminus \Gamma_5$ with the right boundary condition imposed. Although in the presentation of this paper $Z_{CS}^{(a)}(u)$ is defined perturbatively on the cover space parametrized by the logarithmic data u instead of x , $Z_{CS}^{(a)}(u)$ can be defined nonperturbatively as in [76, 77], where it is shown that the nonperturbative $Z_{CS}^{(a)}(u)$ manifestly depends on $x = \exp(u)$. Therefore $Z_{CS}^{(a)}(u)Z_{CS}^{(\bar{a})}(\bar{u})$ depends on the boundary data or spin-network data in a desired manner.

When we generalize our framework from a 4-simplex to a generic simplicial manifold, the class of Chern-Simons 3d blocks $Z_{CS}^{(a)}(u)Z_{CS}^{(\bar{a})}(\bar{u})$, which asymptotically reproduces classical gravity, may ultimately span the physical Hilbert space $\mathcal{H}_{\text{phys}}$ in LQG. The operator constraint equation from the quantizing the Lagrangian subvariety \mathcal{L}_A

$$\hat{\mathbf{A}}_m(\hat{x}, \hat{y}, \hbar)Z_{CS}^{(a)}(u) = 0 \quad (6.5)$$

may relate to the quantization of Hamiltonian constraint equation in LQG [91–93], when the proper boundary condition is implemented.

There is an important perspective that we would like to point out before we conclude this section. In [95], it is suggested that the simplicial 4d geometries corresponds to the dynamical vacua of LQG, namely the solution of critical equations of spinfoam amplitude. In the present work and [20], we have made the correspondence between the simplicial 4d geometry and $\text{SL}(2, \mathbb{C})$ flat connections on the graph complement 3-manifold $S^3 \setminus \Gamma_5$, and we have also shown the solutions of critical equations from $\mathcal{A}[u_{ab}; u_a, v_a]$ give the $\text{SL}(2, \mathbb{C})$ flat connections on $S^3 \setminus \Gamma_5$. Therefore we suggest that the moduli space of LQG dynamical vacua can be embedded into the moduli space $\mathcal{M}_{\text{flat}}(S^3 \setminus \Gamma_5, \text{SL}(2, \mathbb{C}))$, where the image of the embedding map is specified by the boundary condition in Section 2.3. We expect that the dynamical properties of the LQG vacua, including the perturbative behavior of LQG, should be largely controlled by $\text{SL}(2, \mathbb{C})$ Chern-Simons theory.

²⁷ $\text{SL}(2, \mathbb{C})$ Chern-Simons theory with purely imaginary t relates to the quantum Lorentz group with real q [23]. Chern-Simons 3d block with our boundary condition implemented may relate to the spinfoam model defined in [54].

7 Beyond A Single 4-Simplex

The above analysis is mainly about the geometry of a single 4-simplex and its correspondence with flat connection on $S^3 \setminus \Gamma_5$. The analysis can be generalized to arbitrary simplicial geometry in 4-dimensional manifold with arbitrary number of 4-simplices. In this section we give the idea of construction and the results, while the details is given in [77].

A 4-dimensional simplicial complex \mathcal{K}_4 is built by gluing a number of 4-simplices σ . The simplicial geometry on \mathcal{K}_4 is made by the constant curvature geometry on each 4-simplex, with the distributional curvature located at the 2d hinges in the 4-simplex-gluing interface. The simplicial geometries on \mathcal{K}_4 corresponds to a class of $\text{SL}(2, \mathbb{C})$ flat connections on a 3-manifold \mathcal{M}_3 . \mathcal{M}_3 is obtained by gluing N copies $S^3 \setminus \Gamma_5$ (N is the number of 4-simplices in \mathcal{K}_4) as Fig. 8.

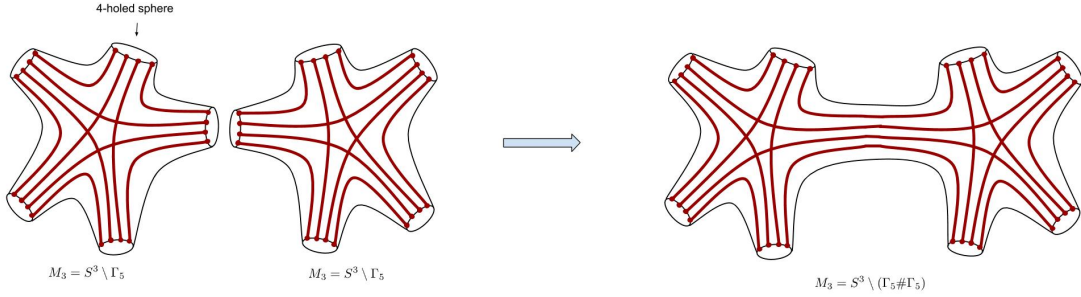


Figure 8. Let's view the graph complement 3-manifold $S^3 \setminus \Gamma_5$ from 4 dimensions, and draw $S^3 \setminus \Gamma_5$ by suppressing 1 dimension. The 3-manifold $S^3 \setminus \Gamma_5$ has five “big boundary” components as five 4-holed spheres, corresponding to five vertices of Γ_5 . $S^3 \setminus \Gamma_5$ also has ten “small boundary” components as ten cylinders, corresponding to ten edges of Γ_5 . The red curves are the tunnels connecting the holes in big boundary components, from removing the tubular neighborhoods of 10 edges. The tunnels give the 10 small boundary components. The union of big and small boundary components gives the closed 2-surface $\Sigma_6 = \partial(S^3 \setminus \Gamma_5)$. 2 graph complement 3-manifolds can be glued through a pair of big boundary components, i.e. a pair of 4-holed spheres, via a certain identification of holes. After the gluing, some of the tunnels are continued from one $S^3 \setminus \Gamma_5$ to the other. Note that in this figure, the properties of crossings are not shown.

The 3-manifold \mathcal{M}_3 can be constructed in the following way (see Fig. 8): Corresponding to gluing a pair of 4-simplices in 4d through a pair of tetrahedra, a 3-manifold is constructed by gluing a pair of $S^3 \setminus \Gamma_5$ through a pair of 4-holed spheres. The boundary Σ_6 of $S^3 \setminus \Gamma_5$ can be decomposed into 2 types of components²⁸, i.e. (a) “big boundary” consisting of five 4-holed spheres from removing the neighborhood of five vertices in Γ_5 , and (b) “small boundary” consisting of 10 cylinders from removing the tubular neighborhood of 10 edges in Γ_5 . When a pair of $S^3 \setminus \Gamma_5$ are glued through a pair of 4-holed spheres via a certain identification of holes, the resulting 3-manifold is a graph complement $S^3 \setminus (\Gamma_5 \# \Gamma_5)$ of a bigger graph. $\Gamma_5 \# \Gamma_5$ is obtained by removing a vertex in each Γ_5 , and connecting the resulting 4 pairs of open edges. By this procedure, we obtain

$$\mathcal{M}_3 = \underbrace{(S^3 \setminus \Gamma_5) \cup \dots \cup (S^3 \setminus \Gamma_5)}_{N \text{ copies}} = \mathfrak{X}_3 \setminus \Gamma_5^{\#N} \quad (7.1)$$

Here \mathfrak{X}_3 is in general a more complicated closed 3-manifold than S^3 . $\pi_1(\mathfrak{X}_3)$ may be nontrivial. It can be seen e.g. when we glue 2 pairs of 4-hole spheres between 2 copies of $S^3 \setminus \Gamma_5$. Here N corresponds to the number of 4-simplices in 4-manifold.

We impose the boundary condition in Section 2.2 to specify the $\text{SL}(2, \mathbb{C})$ flat connection on \mathcal{M}_3 , i.e. the flat connections restricted on big boundary components of \mathcal{M}_3 become $\text{SU}(2)$ flat connections on 4-holed

²⁸They are also called “geodesic boundary” and “generalized cusps”.

spheres. However in addition to the boundary condition, we have to require that on the 4-holed sphere, being interface of the gluing $(S^3 \setminus \Gamma_5)$'s, the $\text{SL}(2, \mathbb{C})$ flat connection has to reduce to $\text{SU}(2)$ as well, in order that the flat connection in each copy of $(S^3 \setminus \Gamma_5)$ determines a constant curvature 4-simplex geometry. Given a $\text{SL}(2, \mathbb{C})$ flat connection on \mathcal{M}_3 satisfying the above requirements, it determines a convex constant curvature 4-simplex geometry from each copies of $S^3 \setminus \Gamma_5$ by Theorem 2.4.

The fundamental group of \mathcal{M}_3 is obtained simply by making product of $\pi_1(S^3 \setminus \Gamma_5)$ and identifying the generators corresponding to the 4-holed spheres being the gluing interface. In terms of holonomies, there may be a parallel transportation between the base points of loops ℓ_{ab} in different copies of $S^3 \setminus \Gamma_5$. Given a pair of glued $S^3 \setminus \Gamma_5$, by the uniqueness in Lemma 2.3, the isomorphisms Eq.(2.36) give 2 identifications between the loops in two copies $S^3 \setminus \Gamma_5$ and the simple paths in two 4-simplices. The isomorphisms induce 2 isomorphisms S_1, S_2 as Eq.(2.24) between the identified loops and the simple paths in two tetrahedra from two 4-simplices. Since the loops are identified, the composed map $S_1 \circ S_2^{-1}$ identifies the simple paths in two tetrahedra. An $\text{SL}(2, \mathbb{C})$ flat connection in \mathcal{M}_3 considered here gives an $\text{SU}(2)$ flat connection on the interface 4-holed sphere, which determines uniquely a convex constant curvature tetrahedron by Theorem 2.1. Such a geometrical constant curvature tetrahedron is shared by the 2 geometrical 4-simplex, since the simple paths of tetrahedra has been identified. Therefore the geometrical 4-simplices determined from all copies of $S^3 \setminus \Gamma_5$ glue geometrically and form a large simplicial geometry. All 4-simplices and tetrahedra have the same constant curvature Λ .

Note that the large simplicial geometry on the simplicial complex is not necessarily constant curvature. It can approximate arbitrary Lorentzian geometry on a 4-dimensional manifold, in the sense of Regge calculus [44].

In a single copy of $S^3 \setminus \Gamma_5$, an $\text{SL}(2, \mathbb{C})$ flat connection A corresponding to 4-simplex geometry is always accompanied by its parity partner \tilde{A} , which determines the same geometry but with different 4d orientation by Theorem 2.5. A, \tilde{A} are related by a complex conjugate with respect to the complex structure of $\text{SL}(2, \mathbb{C})$ i.e. A, \tilde{A} give the same $\text{SU}(2)$ flat connection on 4-holed spheres. On \mathcal{M}_3 from gluing N copies of $S^3 \setminus \Gamma_5$, there are 2^N parity-related flat connections, which determine the same geometry on the simplicial complex. Each of the 2^N flat connections associates with a choice of 2 possible orientations in each individual 4-simplex²⁹. All of the parity-related flat connections give the same set of $\text{SU}(2)$ flat connections on all 4-holed spheres, including the big boundary components and gluing interfaces. Among the 2^N parity-related flat connections, there are only 2 flat connections associated with the 2 possible uniform 4d orientations on the entire simplicial complex, which we call the *global parity pair* and denote again by A, \tilde{A} .

In terms of complex FN coordinate on $\mathcal{M}_{\text{flat}}(\partial\mathcal{M}_3, \text{SL}(2, \mathbb{C}))$, the global parity pair $A, \tilde{A} \in \mathcal{M}_{\text{flat}}(\mathcal{M}_3, \text{SL}(2, \mathbb{C}))$ can be written as

$$A = [x_\ell, y_\ell; x_B, y_B], \quad \tilde{A} = [x_\ell, \tilde{y}_\ell; x_B, y_B] \quad (7.2)$$

where x_ℓ, y_ℓ is the complex length and twist variables of a small boundary component ℓ ³⁰. x_B, y_B is the canonical coordinate of $\mathcal{M}_{\text{flat}}(4\text{-holed sphere}, \text{SU}(2))$ of a big boundary component. Here the variables $[x_\ell; x_B, y_B]$ are treated as the boundary data.

A small boundary component ℓ corresponds to a unique triangle Δ_ℓ in the simplicial complex \mathcal{K}_4 ³¹. x_ℓ relates to the triangle area \mathfrak{a}_ℓ of Δ_ℓ in the same way as before, e.g. in Eq.(4.14). The relation between y_ℓ and hyperdihedral angle is given by a sum of Eq.(3.15) over all the 4-simplices sharing Δ_ℓ , i.e.

$$\begin{aligned} \ln y_\ell = & -\frac{1}{2} \nu \text{sgn}(V_4) \sum_{\sigma, \Delta_\ell \subset \sigma} \Theta_\ell(\sigma) - i\nu \theta_\ell + \frac{\ln \chi_\ell(\xi)}{2} \\ & \text{mod } 2\pi i N_\ell, \quad N_\ell \in \mathbb{Z}, \end{aligned} \quad (7.3)$$

²⁹The same phenomena happens in the asymptotics of LQG spinfoam models [56].

³⁰When the small boundary component ℓ is a torus cusp, x_ℓ, y_ℓ is simply the eigenvalues of meridian and longitude loop holonomies.

³¹ Δ_ℓ is an internal triangle when ℓ is a torus cusp. Δ_ℓ is a boundary triangle when ℓ is a cylinder connecting 2 big boundary components. If the 4-manifold is closed and the simplicial complex doesn't have boundary, the corresponding \mathcal{M}_3 has only torus cusps.

where $\Theta_\ell(\sigma)$ is the hyperdihedral (boost) angle in the 4-simplex σ hinged by Δ_ℓ . $\text{sgn}(V_4)$ is a global sign by the uniform 4d orientation. y_ℓ and \tilde{y}_ℓ relate to two different signs $\text{sgn}(V_4) = \pm 1$ respectively.

We define the logarithmic variables u, v in the same way as before, and choose a canonical lift to the cover space for the boundary data $[x_\ell; x_B, y_B] \mapsto [u_\ell; u_B, v_B]$. We also choose two arbitrary lifts $\alpha, \tilde{\alpha}$ for $y_\ell \mapsto v_\ell^\alpha$ and $\tilde{y}_\ell \mapsto \tilde{v}_\ell^{\tilde{\alpha}}$. The holomorphic 3d block $Z_{CS}^{(\alpha)}(\mathcal{M}_3|u)$ of $\text{SL}(2, \mathbb{C})$ Chern-Simons theory on \mathcal{M}_3 can be constructed in the same way as Eq.(4.11), for (A, α) with the reference $(\tilde{A}, \tilde{\alpha})$. The Liouville 1-form is now given by

$$\vartheta = \sum_\ell v_\ell du_\ell + \sum_B v_B du_B \quad (7.4)$$

The integration contour of $\int_{\mathbb{C}} \vartheta$ is in $\mathcal{L}_A \simeq \mathcal{M}_{\text{flat}}(\mathcal{M}_3, \text{SL}(2, \mathbb{C}))$, which is a holomorphic Lagrangian subvariety in $\mathcal{M}_{\text{flat}}(\partial\mathcal{M}_3, \text{SL}(2, \mathbb{C}))$.

The semiclassical asymptotic behavior of $Z_{CS}^{(\alpha)}(\mathcal{M}_3|u)$ can be analyzed in the same way as in Section 4.2, which leads to the following generalization of Eq.(4.27)

$$\begin{aligned} Z_{CS}^{(\alpha)}(\mathcal{M}_3|u) Z_{CS}^{(\tilde{\alpha})}(\mathcal{M}_3|\tilde{u}) &= \exp \left[\frac{i}{\hbar} 2\text{Re} \left(\frac{\Lambda t}{12\pi i} \right) \left(\sum_\ell \mathbf{a}_\ell \sum_{\sigma, \Delta_\ell \subset \sigma} \Theta_\ell(\sigma) - \Lambda \sum_\sigma \text{Vol}_4^\Lambda(\sigma) \right) \right] \\ &\times \exp \left[\frac{i}{\hbar} 2\text{Re} \left(\frac{\Lambda t}{12\pi i} C_{\tilde{\alpha}}^\alpha \right) + \frac{i}{\hbar} 2\text{Re} \left(\frac{\Lambda t}{6} \right) \sum_\ell \Delta N_\ell \mathbf{a}_\ell + \dots \right]. \end{aligned} \quad (7.5)$$

where the lift-independent term

$$S_{\text{Regge}}^\Lambda = \sum_\ell \mathbf{a}_\ell \sum_{\sigma, \Delta_\ell \subset \sigma} \Theta_\ell(\sigma) - \Lambda \sum_\sigma \text{Vol}_4^\Lambda(\sigma) \quad (7.6)$$

is the Lorentzian Regge action of Einstein gravity on the simplicial complex \mathcal{K}_4 [44–46]. $\sum_{\sigma, \Delta_\ell \subset \sigma} \Theta_\ell(\sigma)$ is the Lorentzian deficit angle when Δ_ℓ is an internal triangle in \mathcal{K}_4 , while it is a hyperdihedral boost angle when Δ_ℓ is a boundary triangle of \mathcal{K}_4 . The gravitational constant G_N is given by Eq.(4.28). $C_{\tilde{\alpha}}^\alpha$ is again a integration constant. $\frac{i}{\hbar} 2\text{Re} \left(\frac{\Lambda t}{6} \right) \sum_\ell \Delta N_\ell \mathbf{a}_\ell$ is lift-dependent and taking discrete values. This term disappears when the quantization condition $2\text{Re} \left(\frac{\Lambda t}{6} \right) \sum_\ell \Delta N_\ell \mathbf{a}_\ell \in 2\pi\hbar\mathbb{Z}$ or $t \in i\mathbb{R}$ is satisfied.

Acknowledgements

MH would like to thank V. V. Fock, J. Mourão, R. Van der Veen, Z. Sun for several fruitful discussions. MH would also like to thank F. Vidotto for the invitation and hospitality for visiting Radboud Universiteit Nijmegen, and for her interesting comments for this work. MH acknowledges the funding received from the People Programme (Marie Curie Actions) of the European Union's Seventh Framework Programme (FP7/2007-2013) under REA grant agreement No. 298786. MH also acknowledges the funding received from Alexander von Humboldt Foundation.

A Proof of Lemma 2.2

Let's consider a flatly embedded 2-surface f in the Lorentzian spacetime $(\mathfrak{M}_4, g_{\alpha\beta})$ with constant curvature Λ . Given the time-like normal u^α and space-like normal n^α of a flatly embedded surface f , we consider the quantity

$$h_\rho^\mu \nabla_\mu [u_\alpha \wedge n_\beta] = 2h_\rho^\mu (\nabla_\mu u_{[\alpha}) n_{\beta]} + 2h_\rho^\mu u_{[\alpha} (\nabla_\mu n_{\beta]}) \quad (\text{A.1})$$

where ∇_μ is the torsion-free covariant derivative satisfying $\nabla_\mu g_{\alpha\beta} = 0$. By using the definition of projection h^α_α and the fact $n^\alpha n_\alpha = 1, u^\alpha u_\alpha = -1$,

$$\begin{aligned} h^\mu_\rho \nabla_\mu n_\beta &= h^\mu_\rho (h^\nu_\beta - u^\nu u_\beta + n^\nu n_\beta) \nabla_\mu n_\nu = h^\mu_\rho (h^\nu_\beta - u^\nu u_\beta) \nabla_\mu n_\nu \\ h^\mu_\rho \nabla_\mu u_\beta &= h^\mu_\rho (h^\nu_\beta - u^\nu u_\beta + n^\nu n_\beta) \nabla_\mu u_\nu = h^\mu_\rho (h^\nu_\beta + n^\nu n_\beta) \nabla_\mu u_\nu. \end{aligned} \quad (\text{A.2})$$

Therefore,

$$h^\mu_\rho \nabla_\mu [u_\alpha \wedge \hat{n}_\beta] = 2h^\mu_\rho h^\nu_{[\alpha} \nabla_\mu u_\nu n_{\beta]} + 2u_{[\alpha} h^\mu_\rho h^\nu_{\beta]} \nabla_\mu n_\nu = 0 \quad (\text{A.3})$$

since 2-surface f is flatly embedded.

The covariant derivative ∇_α relates the spin connection ω_α via $\nabla_\alpha e^\mu_I - \omega_{\alpha I}^J e^\mu_J = \partial_\alpha e^\mu_I + \Gamma_{\alpha\beta}^\mu e^\beta_I - \omega_{\alpha I}^J e^\mu_J = 0$ where e^μ_I is an orthonormal frame on \mathfrak{M}_4 . We define the ‘‘internal’’ normal vectors to the 2-surface by $u^I = e^I_\alpha u^\alpha$ and $n^I = e^I_\alpha n^\alpha$, and define an (internal) covariant derivative by $D_\alpha v^I = \partial_\alpha v^I + \omega_{\alpha I}^J v_J$. Then we have the following relation from Eq.(A.3):

$$h^\alpha_\beta D_\alpha [u^I \wedge n^J] = 0 \quad (\text{A.4})$$

where $u^I \wedge n^J = 2u^{[I} \hat{n}^{J]}$. For any choice of the 2-surface orientation and spacetime orientation, we have the following relation between $u^I \wedge n^J$ and the area density bivector $\varepsilon^{\alpha\beta} e^I_\alpha e^J_\beta$ ($\varepsilon_{\alpha\beta}$ is the area element on f):

$$\frac{1}{2} \varepsilon^{IJ}_{KL} [\varepsilon^{\alpha\beta} e^K_\alpha e^L_\beta] = \pm \frac{1}{2} \varepsilon^{IJ}_{KL} \varepsilon^{\mu\nu\alpha\beta} u_\mu n_\nu e^K_\alpha e^L_\beta = \pm \frac{1}{2} \varepsilon^{IJKL} \varepsilon_{MNKL} u^M n^N = \mp 2u^{[I} n^{J]} \quad (\text{A.5})$$

where \pm comes from the choices of 2-surface orientation and spacetime orientation. Since $D_\alpha \varepsilon_{IJKL} = 0$, Eq.(A.4) is equivalent to

$$h^\mu_\nu D_\mu [\varepsilon^{\alpha\beta} e^I_\alpha e^J_\beta] = 0. \quad (\text{A.6})$$

The area density bivector $\varepsilon^{\alpha\beta} e^I_\alpha e^J_\beta$ can be decomposed into the self-dual and anti-self-dual parts:

$$\varepsilon^{\alpha\beta} e^I_\alpha e^J_\beta = B_+^i T_+^i + B_-^i T_-^i, \quad (i = 1, 2, 3) \quad (\text{A.7})$$

where B_+ and B_- are related by complex conjugate. Using spinor we can write $[\varepsilon^{\alpha\beta} e_\alpha e_\beta]_+ = B_+ \cdot \vec{\tau}^{32}$. The parallel transportation equation Eq.(A.6) can be integrated and give the following relation between the area density bivectors at the base point $O \in \partial f$ and at an arbitrary point x on the 2-surface f

$$U(x)^{-1} [\varepsilon^{\alpha\beta} e_\alpha e_\beta]_+ (O) U(x) = [\varepsilon^{\alpha\beta} e_\alpha e_\beta]_+ (x). \quad (\text{A.8})$$

where $U(x)$ is the $\text{SL}(2, \mathbb{C})$ holonomy along any path in f oriented from x to O .

The 2-form restricted on the 2-surface $e^I \wedge e^J = [\varepsilon^{\alpha\beta} e^I_\alpha e^J_\beta] \underline{\varepsilon}$ where $\underline{\varepsilon}$ is the area element on f . Since $(\mathfrak{M}_4, g_{\alpha\beta})$ is constant curvature, we have $\mathcal{R}^{IJ}(\omega) = \frac{\Lambda}{3} e^I \wedge e^J$ in \mathfrak{M}_4 . Restrict onto the 2-surface f :

$$\mathcal{R}^{IJ}(\omega) = \frac{\Lambda}{3} [\varepsilon^{\alpha\beta} e^I_\alpha e^J_\beta] \underline{\varepsilon} \quad (\text{A.9})$$

Combining with the above parallel transportation Eq.(A.8), the self-dual part of this relation gives

$$U(x) R(x) U(x)^{-1} = \frac{\Lambda}{3} [\varepsilon^{\alpha\beta} e_\alpha e_\beta]_+ (O) \underline{\varepsilon}(x). \quad (\text{A.10})$$

³² In terms of Weyl spinor indices (in the convenient of [96]), $[\varepsilon^{\alpha\beta} e_\alpha e_\beta] = B_+^{AB} \varepsilon^{A'B'} + B_-^{A'B'} \varepsilon^{AB}$ where $B_+^{AB} = B_+^{(AB)}$ is the self-dual part of $[\varepsilon^{ab} e_a e_b]$ in spinor representation, and $B_-^{A'B'} = \overline{B_+^{AB}}$. Using $\text{su}(2)$ basis $(B_+)^A_B = B_+ \cdot \vec{\tau}^A_B$ (ε_{AB} is used for lowering and raising spinor indices). Eq.(A.8) may be written as $U(x)^A_C U(x)^B_D B_+^{CD}(x) = B_+^{AB}(O) (U(x)^A_C \in \text{SL}(2, \mathbb{C}))$, or by lower and raising indices $U(x)^A_C U(x)^D_B (B_+)^C_D(x) = -(B_+)^A_B(O)$. However $U(x)^D_B = \varepsilon_{AB} \varepsilon^{DC} U(x)^A_C$ which results in $U(x)^A_B U(x)^B_C = -\delta^A_C (\varepsilon^{AB} \varepsilon_{BC} = -\delta^A_C$ in our convention), i.e. $U(x)^B_C = -[U(x)^{-1}]^B_C$. Therefore we have $U(x)^A_C (B_+)^C_D(x) [U(x)^{-1}]^D_B = (B_+)^A_B(O)$, whose infinitesimal version is given by $D_\alpha B_+^{AB} = \partial_\alpha B_+^{AB} + \omega_{\alpha C}^A B_+^{CB} + \omega_{\alpha C}^B B_+^{AC} = 0$. ω_{α}^{AB} is the spinor representation of ω_{α}^{IJ} .

where the $\mathfrak{sl}_2\mathbb{C}$ curvature R is the self-dual part of $\mathcal{R}^{IJ}(\omega)$. The anti-self-dual part of the relation is given by simply complex conjugate.

Let's briefly recall the nonabelian stokes theorem [97]: Given a smooth connection ω on a manifold and an embedded 2-surface f with boundary, the loop holonomy $U_{\partial f}(\omega)$ relates to the $\mathfrak{sl}_2\mathbb{C}$ curvature $R = d\omega + \omega \wedge \omega$ by the following relation:

$$U_{\partial f}(\omega) = \mathcal{P} \exp \left\{ \int_f U(x) R(x) U(x)^{-1} \right\} := \lim_{\delta f(x) \rightarrow x} \mathcal{P} \prod_x \exp \left\{ U(x) \left[\int_{\delta f(x)} R \right] U(x)^{-1} \right\}. \quad (\text{A.11})$$

Here a partition of 2-surface f is defined, where each plaquette $\delta f(x)$ is associated with a base point x . Moreover the discretized 2-surface f is equipped with (1) a face ordering \mathcal{P} as an ordering of the plaquettes in f , and (2) A path system such that each base point x is connected to the base point O of the holonomy $U_{\partial f}(\omega)$ by a unique path $p_{O,x}$. $U(x)$ is the holonomy along the path from x to O . An example of the choice of face ordering and path system can be found in [97]. The above relation holds in the continuum limit $\lim_{\delta f(x) \rightarrow x}$ as each plaquette $\delta f(x)$ shrinks into a point x .

By Eq.(A.10), we have for each plaquette $U(x) \left[\int_{\delta f(x)} R \right] U(x)^{-1} = \frac{\Lambda}{3} \delta a(x) [\varepsilon^{\alpha\beta} e_\alpha e_\beta]_+(O)$, where $\delta a(x)$ is the area of the plaquette $\delta f(x)$. The above quantity is commutative between different x . Thus the face ordering \mathcal{P} in nonabelian stokes theorem becomes trivial. As a result,

$$U_{\partial f}(\omega) = \exp \left\{ \frac{\Lambda}{3} a [\varepsilon^{\alpha\beta} e_\alpha e_\beta]_+(O) \right\}, \quad (\text{A.12})$$

where $\sum_x \delta a(x) = a$ is the area of the 2-surface f .

We make a partial gauge-fixing such that e_0^α is the time-like normal vector of the surface f at the base point O . Then $[\varepsilon^{\alpha\beta} e_\alpha e_\beta]_+(O)$ belongs to the \mathfrak{su}_2 sub-algebra of $\mathfrak{sl}_2\mathbb{C}$. So in this gauge

$$U_{\partial f}(\omega) = \exp \left\{ \frac{\Lambda}{3} a \hat{n} \cdot \vec{\tau} \right\} \in \text{SU}(2), \quad (\text{A.13})$$

Here \hat{n} , being a unit 3-vector in \mathbb{R}^3 , is a spatial normal vector of f , because in this gauge $n_0 = n_\alpha e_0^\alpha = 0$ and

$$\begin{aligned} \hat{n}^k &:= [\varepsilon^{\alpha\beta} e_\alpha e_\beta]_+^k(O) = \frac{1}{2} \varepsilon^{0ijk} [\varepsilon^{\alpha\beta} e_\alpha e_\beta]_{ij}(O) + i [\varepsilon^{ab} e_a e_b]^{0k}(O) \\ &= \mp \frac{1}{2} \varepsilon^{IJLK} u_I \varepsilon^{\alpha\beta\mu\nu} u_\alpha n_\beta e_{\mu J} e_{\nu L}(O) = \mp \frac{1}{2} \varepsilon^{IJLK} u_I \varepsilon_{MNJL} u^M n^N(O) = \mp n^k. \end{aligned} \quad (\text{A.14})$$

where \mp comes from the choices of 2-surface orientation and spacetime orientation.

B Dihedral Angles and Boundary Data

In this appendix, we compute the dihedral angle between any pair of 2-surfaces in the geometrical 4-simplex determined by the flat connection on $S^3 \setminus \Gamma_5$, and show that all the 10 dihedral angles coincide with the ones determined by the boundary data in Section 2.1. Therefore the tetrahedra on the boundary of 4-simplex are congruent with the tetrahedra resulting from the boundary data.

All the dihedral angles between surfaces intersecting at $\bar{1}$ have been determined in Eq.(??) (minus sign is ruled out). Let's consider the hypersurface \mathfrak{S}_1 intersecting other hypersurfaces at the vertex $\bar{3}$. Given the relations $v_1 \mathcal{E}_{1a}(\bar{3}) = v_a \mathcal{E}_{a1}(\bar{3}) \equiv v \tilde{\mathcal{E}}_{1a}(\bar{3})$ ($a \in \{2, 4, 5\}$) and that $\Omega[\bar{1}, \bar{3}]$ is the holonomy of spin connection along the intersection edge $(\bar{1}, \bar{3})$. There may be freedom in $\Omega[\bar{1}, \bar{3}]$, which may be viewed as the freedom of the position of hypersurface \mathfrak{S}_1 . But the dihedral angles between a pair of 2-surfaces Δ_{ae} and Δ_{ab} ($a, b, e \in \{1, 2, 4, 5\}$) adjacent to the vertex $\bar{3}$ can be computed unambiguously: A surface Δ_{1a} ($a = 2, 4, 5$), obtained by the intersection between \mathfrak{S}_1 and $\mathfrak{S}_{a=2,4,5}$, is adjacent to the $\bar{3}$ but nonadjacent to $\bar{1}$. By the parallel transportation $\hat{\Omega}[\bar{1}, \bar{3}]$ ($\hat{g} \in \text{SO}^+(1, 3)$ for any $g \in \text{SL}(2, \mathbb{C})$) denotes the vector representation of g)

$$\hat{\Omega}[\bar{3}, \bar{1}] [\star \tilde{\mathcal{E}}_{1a}(\bar{1})] \hat{\Omega}[\bar{3}, \bar{1}]^{-1} = \star \tilde{\mathcal{E}}_{1a}(\bar{3}) \quad \text{where} \quad \star \tilde{\mathcal{E}}_{1a}(\bar{1}) = v \hat{n}_{1a}(\bar{1}) \wedge N_1(\bar{1}). \quad (\text{B.1})$$

We have the following relations for Δ_{ab} adjacent to both the vertices 1 and 3 since the surfaces are flatly embedded:

$$\hat{\Omega}[\bar{3}, \bar{1}][\star \tilde{\mathcal{E}}_{ab}(\bar{1})]\hat{\Omega}[\bar{3}, \bar{1}]^{-1} = \star \tilde{\mathcal{E}}_{ab}(\bar{3}) \quad (\text{B.2})$$

Therefore for all the surfaces Δ_{ae} adjacent to $\bar{3}$, their area density bivectors can be written as

$$\star \tilde{\mathcal{E}}_{ae}(\bar{3}) = v\hat{n}_{ae}(\bar{3}) \wedge N_a(\bar{3}), \quad \forall a, e \in \{1, 2, 4, 5\} \quad (\text{B.3})$$

where $\hat{n}_{ae}(\bar{3}) = \hat{\Omega}[\bar{3}, \bar{1}]\hat{n}_{ae}(\bar{1}) = \hat{\Omega}[\bar{3}, \bar{1}]\hat{g}_a \hat{n}_{ae}$ and $N_a(\bar{3}) = \hat{\Omega}[\bar{3}, \bar{1}]N_a(\bar{1}) = \hat{\Omega}[\bar{3}, \bar{1}]\hat{g}_a u$. The dihedral angles between 2 surfaces Δ_{ab}, Δ_{ae} can be computed by ($\hat{n}_{ab} = v_a \hat{n}_{ab}$)

$$\cos \phi_{\Delta_{ab}, \Delta_{ae}} = \eta_{IJ} \hat{n}_{ab}^I(\bar{3}) \hat{n}_{ae}^J(\bar{3}) = \hat{n}_{ab} \cdot \hat{n}_{ae}, \quad \forall a, b, e \in \{1, 2, 4, 5\} \quad (\text{B.4})$$

since Δ_{ab}, Δ_{ae} have the same time-like normal $N_a(\bar{3})$ at the vertex 3. The dihedral angles at the $\bar{3}$ uniquely determine all the faces angles adjacent to $\bar{3}$. By the simple expression of $\cos \phi_{\Delta_{ab}, \Delta_{ae}}$ ($a, b, e \in \{1, 2, 4, 5\}$), all the dihedral angles and face angles at $\bar{3}$ are identical to the tetrahedron geometry from boundary data, independent of $\Omega[\bar{1}, \bar{3}]$.

The dihedral angles $\phi_{\Delta_{ab}, \Delta_{ae}}$ hinged by $(\bar{2}, \bar{4})$, $(\bar{2}, \bar{5})$, and $(\bar{4}, \bar{5})$ are the ones which haven't been computed so far. A few dihedral angles hinged by the edges $(\bar{2}, \bar{5})$ and $(\bar{4}, \bar{5})$ can be determined by using the parallel transportations $\Omega[\bar{1}, \bar{5}], \Omega[\bar{3}, \bar{5}]$ of spin connection. They are the dihedral angles in tetra_1 and tetra_3 hinged by $(\bar{2}, \bar{5})$ and $(\bar{4}, \bar{5})$, which can be computed in the same way as above by the parallel transportations $\Omega[\bar{1}, \bar{5}], \Omega[\bar{3}, \bar{5}]$:

$$\begin{aligned} \cos \phi_{\Delta_{13}, \Delta_{12}} &= \hat{n}_{13} \cdot \hat{n}_{12}, & \cos \phi_{\Delta_{31}, \Delta_{32}} &= \hat{n}_{31} \cdot \hat{n}_{32}, \\ \cos \phi_{\Delta_{13}, \Delta_{14}} &= \hat{n}_{13} \cdot \hat{n}_{12}, & \cos \phi_{\Delta_{31}, \Delta_{34}} &= \hat{n}_{31} \cdot \hat{n}_{34}. \end{aligned} \quad (\text{B.5})$$

Again these dihedral angles are identical to the tetrahedron geometries from boundary data.

There are still 5 dihedral angles which haven't been determined yet. Each of these dihedral angles is located in one of the 5 tetrahedra in the boundary of the geometrical 4-simplex. They are (A) $\phi_{\Delta_{13}, \Delta_{15}}$ in tetra_1 , (B) $\phi_{\Delta_{23}, \Delta_{21}}$ in tetra_2 , (C) $\phi_{\Delta_{31}, \Delta_{35}}$ in tetra_3 , (D) $\phi_{\Delta_{41}, \Delta_{43}}$ in tetra_4 , and (E) $\phi_{\Delta_{51}, \Delta_{53}}$ in tetra_5 .

Firstly, the missing dihedral angle $\phi_{\Delta_{23}, \Delta_{21}}$ in tetra_2 can be computed by knowing $\tilde{H}_{24} = \pm \Omega[\bar{1}, \bar{3}]\Omega[\bar{3}, \bar{5}]\Omega[\bar{5}, \bar{1}]$ with $\Omega[\bar{a}, \bar{b}]$ being the parallel transportation of spin connection along the edge (\bar{a}, \bar{b}) : At the vertex $\bar{1}$, we have

$$\begin{aligned} \star \tilde{\mathcal{E}}_{23}(\bar{1}) &= \hat{g}_2(v\hat{n}_{23} \wedge u)\hat{g}_2^{-1} = v\hat{n}_{23}(\bar{1}) \wedge N_2(\bar{1}), \\ \star \tilde{\mathcal{E}}_{21}(\bar{1}) &= \hat{g}_2(v\hat{n}_{23} \wedge u)\hat{g}_2^{-1} = v\hat{n}_{21}(\bar{1}) \wedge N_2(\bar{1}), \end{aligned} \quad (\text{B.6})$$

with $N_2(\bar{1}) = \hat{g}_2 u$ and $\hat{n}_{23,21}(\bar{1}) = \hat{g}_2 \hat{n}_{23,21}$. We parallel transport both $\star \tilde{\mathcal{E}}_{21}(\bar{1})$ and $\star \tilde{\mathcal{E}}_{23}(\bar{1})$ to the vertex $\bar{5}$ by $\Omega[\bar{5}, \bar{3}]\Omega[\bar{3}, \bar{1}]$ and $\Omega[\bar{5}, \bar{1}]$: ($\tilde{H}_{24} = g_2 H_{24} \hat{g}_2^{-1}$ where $H_{24} \in \text{SU}(2)$)

$$\begin{aligned} \hat{\Omega}[\bar{5}, \bar{3}]\hat{\Omega}[\bar{3}, \bar{1}][\star \tilde{\mathcal{E}}_{21}(\bar{1})]\hat{\Omega}[\bar{3}, \bar{1}]^{-1}\hat{\Omega}[\bar{5}, \bar{3}]^{-1} &= \hat{\Omega}[\bar{5}, \bar{1}]\hat{H}_{24}^{-1}[v\hat{n}_{21}(\bar{1}) \wedge N_2(\bar{1})]\hat{H}_{24}\hat{\Omega}[\bar{5}, \bar{1}]^{-1} \\ &= \hat{\Omega}[\bar{5}, \bar{1}][v(\hat{g}_2 \hat{H}_{24}^{-1} \hat{n}_{21}) \wedge N_2(\bar{1})]\hat{\Omega}[\bar{5}, \bar{1}]^{-1} \\ \hat{\Omega}[\bar{5}, \bar{1}][\star \tilde{\mathcal{E}}_{23}(\bar{1})]\hat{\Omega}[\bar{5}, \bar{1}]^{-1} &= \hat{\Omega}[\bar{5}, \bar{1}][v\hat{g}_2 \hat{n}_{23} \wedge N_2(\bar{1})]\hat{\Omega}[\bar{5}, \bar{1}]^{-1}. \end{aligned} \quad (\text{B.7})$$

Both Δ_{23}, Δ_{21} share the same time-like normal vector $\hat{\Omega}[\bar{5}, \bar{1}]N_2(1)$ at $\bar{5}$. Thus the dihedral angle $\phi_{\Delta_{23}, \Delta_{21}}$ is given by

$$\cos \phi_{\Delta_{23}, \Delta_{21}} = (\hat{H}_{24}^{-1} \hat{n}_{21}) \cdot \hat{n}_{23}, \quad (\text{B.8})$$

which is identical to the corresponding tetrahedron geometry in the boundary data (recall Eq.(??)). All the dihedral angles of tetra_2 has been shown to be identical to the corresponding tetrahedron geometry in the

boundary data. Therefore tetra_2 in the geometrical 4-simplex is congruent to the corresponding tetrahedron from the boundary data since they have the same Gram matrix.

In exactly the same way, the last dihedral angle $\phi_{\Delta_{41}, \Delta_{43}}$ in tetra_4 can also be computed by using $\tilde{H}_{24} = \pm \Omega[\bar{1}, \bar{3}] \Omega[\bar{3}, \bar{5}] \Omega[\bar{5}, \bar{1}]$. The result is

$$\cos \phi_{\Delta_{41}, \Delta_{43}} = (\hat{H}_{24}^{-1} \hat{n}_{41}) \cdot \hat{n}_{43}. \quad (\text{B.9})$$

Thus tetra_4 in the geometrical 4-simplex is congruent to the corresponding tetrahedron from the boundary data. The rest 3 dihedral angles are computed in the same way as the above, which shows that the $\text{tetra}_{1,3,5}$ in the geometrical 4-simplex are congruent to the corresponding tetrahedra from the boundary data.

C Poisson Bracket between Complex FN Variables

To simplify the notation in the expression of FN twist variable τ in Eq.(3.7), we make the short-hand notations for the flat sections in the region we are interested in

$$s_{ab} \equiv s, \quad s_{bd} \equiv s_1, \quad s_{bh} \equiv s'_1, \quad s_{ac} \equiv s_2, \quad s_{ae} \equiv s'_2 \quad (\text{C.1})$$

The holonomy along the meridian cycle γ_x in Fig. 9 corresponding to H_{ab} (up to conjugation), which we denote by H here. The section s is the eigenvector of H with eigenvalue x . $s_{1,2}$ and $s'_{1,2}$ are the sections for the eigenvectors for the meridian holonomies around other holes, shown in Fig. 9. The complex FN twist variable is written as

$$\tau = - \frac{\langle s_1 \wedge s'_1 \rangle}{\langle s_1 \wedge s \rangle \langle s'_1 \wedge s \rangle} \frac{\langle s_2 \wedge s \rangle \langle s'_2 \wedge s \rangle}{\langle s_2 \wedge s'_2 \rangle}. \quad (\text{C.2})$$

We draw a longitude curve γ_τ as in Fig. 9, whose source and target are labeled by 1 and 2. The target point 2 is such that s evaluated at 2, $s(2) = d$, is the initial value of the section (the framing flag). The holonomy along γ_τ is denoted by G . γ_τ intersects γ_x at a single point p .

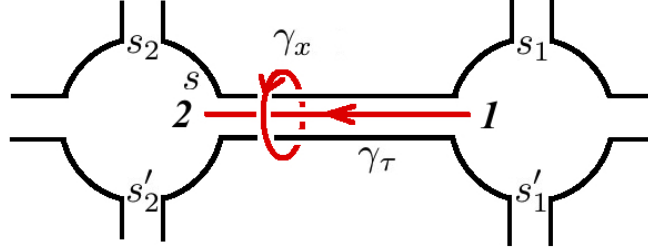


Figure 9. Comparing Fig. 5, the 4-holed sphere on the right is S_b , and the 4-holed sphere on the left is S_a .

The brackets $\langle v_1 \wedge v_2 \rangle$ is $\text{SL}(2, \mathbb{C})$ invariant. We choose to evaluate all the brackets in τ involving s at the point 2, while we evaluate $\langle s_i \wedge s'_i \rangle$ at $i = 1, 2$.

$$\tau = - \frac{\langle s_1(1) \wedge s'_1(1) \rangle}{\langle s_1(2) \wedge d \rangle \langle s'_1(2) \wedge d \rangle} \frac{\langle s_2(2) \wedge d \rangle \langle s'_2(2) \wedge d \rangle}{\langle s_2(2) \wedge s'_2(2) \rangle}. \quad (\text{C.3})$$

where e.g. $\langle s_1(2) \wedge d \rangle = \langle Gs_1(1) \wedge d \rangle$ involves the longitude holonomy.

The Poisson bracket from $\text{SL}(2, \mathbb{C})$ Chern-Simons theory is $\{A_\mu^i(x), A_\nu^j(x')\} = (-\frac{8\pi}{t})\varepsilon_{\mu\nu}\delta^{ij}\delta^{(2)}(x, x')$, where the Lie algebra basis is chosen to be $\tau_j = -\frac{i}{2}\sigma_j$. The Poisson bracket between the meridian and longitude holonomies H and G can be computed

$$\{G \otimes H\} = \frac{2\pi}{t} (G[2, p]\sigma_j G[p, 1]) \otimes (H'[*] \sigma_j H[p, *]). \quad (\text{C.4})$$

The intersection point p have split H and G into $G = G[2, p]G[p, 1]$ and $H = H'[*]H[p, *]$, where $*$ is the base point of H . At the intersection p , the intersection number between γ_x and γ_τ is $\varepsilon_{\mu\nu}\dot{\gamma}_\tau^\mu(p)\dot{\gamma}_x^\nu(p) = 1$. The Poisson bracket between a pair of holonomies is nonvanishing only when they have nontrivial intersection number. Therefore the contribution of $\{\tau, H\}$ comes from $\langle s_1(2) \wedge d \rangle \langle s'_1(2) \wedge d \rangle$ in τ .

We take the trace of H and compute

$$\begin{aligned} \langle \langle G s_1(1) \wedge d \rangle, \text{tr} H \rangle &= \frac{2\pi}{t} \text{tr}(\sigma_j H(p)) \langle G[2, p]\sigma_j G[p, 1] s_1(1) \wedge d \rangle \\ &= \frac{2\pi}{t} \text{tr}(\sigma_j H(p)) \langle \sigma_j s_1(p) \wedge s(p) \rangle \end{aligned} \quad (\text{C.5})$$

where $H(p)$ is the meridian holonomy based at p . We have denote $s_1(p) = G[p, 1]s_1(1)$ and $s(p) = G[p, 2]d$. Generically the matrix $H(p)$ can be diagonalized by $H(p) = M \text{diag}(x, x^{-1}) M^{-1}$ where $M \in \text{SL}(2, \mathbb{C})$. By using the relation $\sigma_j \otimes \sigma_j = M \sigma_j M^{-1} \otimes M \sigma_j M^{-1}$, we have

$$\begin{aligned} \langle \langle G s_1(1) \wedge d \rangle, \text{tr} H \rangle &= \frac{2\pi}{t} \text{tr}(\sigma_j \text{diag}(x, x^{-1})) \langle M \sigma_j M^{-1} s_1(p) \wedge s(p) \rangle \\ &= \frac{2\pi}{t} (x - x^{-1}) \langle \sigma_3 M^{-1} s_1(p) \wedge M^{-1} s(p) \rangle \end{aligned} \quad (\text{C.6})$$

Moreover $M^{-1}s(p) = (1, 0)^T$ because $s(p)$ is the eigenvector of $H(p)$ with eigenvalue x . By the relation $\varepsilon_{AB}(\sigma_3)^A_C (\varepsilon_3)^B_D = -\varepsilon_{AB}$, we obtain

$$\langle \langle G s_1(1) \wedge d \rangle, \text{tr} H \rangle = -\frac{2\pi}{t} (x - x^{-1}) \langle s_1(p) \wedge s(p) \rangle \quad (\text{C.7})$$

Since $\text{tr} H = x + x^{-1}$, it implies

$$\{ \ln \langle s_1 \wedge s \rangle, \ln x \} = -\frac{2\pi}{t} \quad (\text{C.8})$$

The same computation can be done for the factor $\langle s'_1 \wedge s \rangle$ in τ , while all other factors are Poisson commuting with x . Therefore

$$\{ \ln \tau, \ln x \} = \frac{4\pi}{t}. \quad (\text{C.9})$$

D Hyper-dihedral Boost

In this appendix we present a derivation of Eq.(3.15). The derivation here for a constant curvature 4-simplex is a generalization from the analysis in [55] for a flat 4-simplex. We parametrize $\lambda_{ab} = \exp(-\psi_{ab} - i\theta_{ab})$ where $\psi_{ab}, \theta_{ab} \in \mathbb{R}$. It can be derived from Eq.(3.5) that

$$G_{ab} G_{ab}^\dagger = e^{-4\psi_{ab}(\vec{K} \cdot \hat{n}_{ab})} \quad (\text{D.1})$$

where $\vec{K} \cdot \hat{n}_{ab} \in \mathfrak{sl}_2 \mathbb{C}$ corresponds to a boost generator of Lorentz transformation, satisfying $(\vec{K} \cdot \hat{n}_{ab}) \xi_{ab} = \frac{1}{2} \xi_{ab}$.

The computation of hyperdihedral angles Eqs.(??), (??), and (??) suggests a factorization of each $\hat{G}_{ab} \in \text{SO}^+(1, 3)$

$$\hat{G}_{ab} = \hat{X}_a(a, b)^{-1} \hat{X}_b(a, b). \quad (\text{D.2})$$

$X_a(a, b)$ is a parallel transportation from the local frame $e_I(a)$ adapted to tetra_a to a generic frame located at a vertex of Δ_{ab} . For instance, $X_3(1, 3) = \hat{\Omega}[\bar{5}, \bar{1}]\hat{g}_3$ and $X_1(1, 3) = \hat{\Omega}[\bar{5}, \bar{3}]\hat{\Omega}[\bar{3}, \bar{1}]\hat{g}_1$ parallel transport to the vertex $\bar{5}$ of Δ_{13} .

We consider an arbitrary lift for each $X_a(a, b) \in \text{SL}(2, \mathbb{C})$ of $\hat{X}_a(a, b) \in \text{SO}^+(1, 3)$, and insert in Eq.(D.1) which is lift-independent. Then we have

$$D_{ab}X_a(a, b)X_a(a, b)^\dagger D_{ab}^\dagger = X_b(a, b)X_b(a, b)^\dagger, \quad D_{ab} = X_a(a, b)e^{-2\psi_{ab}(\vec{K} \cdot \hat{n}_{ab})}X_a(a, b)^{-1}. \quad (\text{D.3})$$

The vector representation of the above relation is

$$\hat{D}_{ab}N_a(\Delta_{ab}) = N_b(\Delta_{ab}) \quad (\text{D.4})$$

$N_a(\Delta_{ab}) = \hat{X}_a(a, b) \triangleright (1, 0, 0, 0)^T$ denotes the future-pointing time-like normal of tetra_a located at one of the vertices of Δ_{ab} . For instance $N_1(\Delta_{13}) = N_1(\bar{5})$, $N_3(\Delta_{13}) = N_3(\bar{5})$. \hat{D}_{ab} is a pure boost with the following expression³³:

$$\hat{D}_{ab} = \exp\left(|\Theta_{ab}| \frac{N_a(\Delta_{ab}) \wedge N_b(\Delta_{ab})}{|\star N_a(\Delta_{ab}) \wedge N_b(\Delta_{ab})|}\right) = \exp\left(-\Theta_{ab} \frac{U_a(\Delta_{ab}) \wedge U_b(\Delta_{ab})}{|\star U_a(\Delta_{ab}) \wedge U_b(\Delta_{ab})|}\right) \quad (\text{D.5})$$

$U_a(\Delta_{ab}) = \pm N_a(\Delta_{ab})$ are 5 outward-pointing time-like normal of tetra_a , so some of $U_a(\Delta_{ab})$ are past-pointing. The hyperdihedral (boost) angle Θ_{ab} is defined to be negative for Δ_{ab} with both $U_a(\Delta_{ab})$ and $U_b(\Delta_{ab})$ future-pointing or past-pointing. Θ_{ab} is defined to be positive when one of $U_a(\Delta_{ab})$ is future-pointing and the other $U_b(\Delta_{ab})$ is past-pointing. Θ_{ab} defined in this way is indeed natural and is the one entering the Schläfli identity and Regge action [45, 79].

Let's consider a flatly embedded triangle Δ_{ab} in the constant curvature spacetime $(\mathfrak{M}_4, g_{\alpha\beta})$. $U_a(\Delta_{ab}), U_b(\Delta_{ab})$ are the normal vectors located at a vertex O of Δ_{ab} . We have the following relation:

$$\frac{\star U_a(\Delta_{ab}) \wedge U_b(\Delta_{ab})}{|\star U_a(\Delta_{ab}) \wedge U_b(\Delta_{ab})|} = \text{sgn}(V_4) \left[\varepsilon^{\alpha\beta} e_\alpha e_\beta \right]_{ab}(O), \quad (\text{D.6})$$

where the volume element of Δ_{ab} is defined by $\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta\mu\nu} U_a(\Delta_{ab})^\mu U_b(\Delta_{ab})^\nu$ with $U_a(\Delta_{ab})^\mu = U_a(\Delta_{ab})^I e_I^\mu(O)$. $e_I(O)$ is a generic orthonormal frame at O . $\varepsilon_{\alpha\beta\mu\nu}$ is an arbitrary choice of volume element on \mathfrak{M}_4 compatible with $g_{\alpha\beta}$. $\text{sgn}(V_4)$ is defined by the compatibility between $\varepsilon_{\alpha\beta\mu\nu}$ and e_α^I

$$\varepsilon = \text{sgn}(V_4) e^0 \wedge e^1 \wedge e^2 \wedge e^3. \quad (\text{D.7})$$

When the orthonormal frame e_I^α is given on $(\mathfrak{M}_4, g_{\alpha\beta})$, $\text{sgn}(V_4)$ labels the choices of the orientation of \mathfrak{M}_4 .

In the frame $e_I(a)$ adapted to tetra_a whose time-like normal is $(1, 0, 0, 0)$, the self-dual part $\left[\varepsilon^{\alpha\beta} e_\alpha e_\beta \right]_+$ is identified with $\hat{n} \cdot \vec{\tau} = \hat{n} \cdot \vec{J} \in \mathfrak{sl}_2\mathbb{C}$ in Lemma 2.2. $\mathfrak{sl}_2\mathbb{C}$ is viewed as a 6-dimensional real Lie algebra with generator $\vec{J} := \vec{\tau}$ and $\vec{K} := -i\vec{\tau}$. The duality map \star acts as $\star \vec{J} = -\vec{K}$ and $\star \vec{K} = \vec{J}$. Therefore in the frame $e_I(a)$

$$\star \left[\varepsilon^{\alpha\beta} e_\alpha e_\beta \right]_+ = -\hat{n} \cdot \vec{K} = -\nu \hat{n} \cdot \vec{K}, \quad (\text{D.8})$$

or as a bivector,

$$\star \left[\varepsilon^{\alpha\beta} e_\alpha^I e_\beta^J \right]_{ab} = \nu (\hat{n}_{ab} \wedge u)^{IJ}, \quad u = (1, 0, 0, 0)^T. \quad (\text{D.9})$$

Here $\nu = \text{sgn}(\Lambda)$ is the sign appearing in translating the flat connection data to geometrical data in Section 2.1. As a result from the action of $\hat{X}_a(a, b)$ transforming from $e_I(a)$ to $e_I(O)$,

$$\hat{D}_{ab} = \hat{X}_a(a, b) e^{-\nu \text{sgn}(V_4) \Theta_{ab}(\hat{n}_{ab} \cdot \vec{K})} \hat{X}_a(a, b)^{-1}. \quad (\text{D.10})$$

Comparing with Eq.(D.3) gives

$$\psi_{ab} = \frac{1}{2} \nu \text{sgn}(V_4) \Theta_{ab}. \quad (\text{D.11})$$

³³The norm of the Lorentz bivector X^{IJ} is defined by $|X| = \sqrt{\frac{1}{2} X^{IJ} X_{IJ}}$.

E K_2 -Lagrangian Subvariety

In this appendix, we provide a very brief introduction to the notion of K_2 -Lagrangian subvariety, and explain its relation to the quantizability. The discussion here follows [66]. See also [33, 71] for more detailed discussions.

Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, we define the Abelian group $\mathbb{C}^* \wedge \mathbb{C}^* = \wedge^2 \mathbb{C}$ generated by $a \wedge b$, $a, b \in \mathbb{C}^*$ with the relations

$$a \wedge b = -b \wedge a, \quad (ab) \wedge c = a \wedge c + b \wedge c. \quad (\text{E.1})$$

Let \mathcal{M} be a complex variety, and $\mathbb{C}^*(U_\alpha)$ denotes the set of holomorphic functions $U_\alpha \rightarrow \mathbb{C}^*$ on the chart U_α . A K_2 -symplectic structure on \mathcal{M} is an element $\omega_\alpha^K \in \mathbb{C}^*(U_\alpha) \wedge \mathbb{C}^*(U_\alpha)$ on every coordinate chart U_α , such that on $U_\alpha \cap U_\beta$, $\omega_\alpha^K - \omega_\beta^K = \sum_I z_I \wedge (1 - z_I)$ for some $z_I \in \mathbb{C}^*(U_\alpha \cap U_\beta)$. In other words, A K_2 -symplectic structure on \mathcal{M} belongs to the group $K_2(\mathbb{C})$ being the quotient of $\mathbb{C}^* \wedge \mathbb{C}^*$ by the subgroup generated by $z \wedge (1 - z)$

We define a map $d \ln \wedge d \ln$ from $\mathbb{C}^* \wedge \mathbb{C}^*$ to the space of holomorphic 2-forms $\Omega^2(\mathbb{C})$ by

$$d \ln \wedge d \ln : x \wedge y \mapsto d \ln x \wedge d \ln y \quad (\text{E.2})$$

It is easy to see that $d \ln \wedge d \ln$ is essentially a map from $K_2(\mathbb{C}) \rightarrow \Omega^2(\mathbb{C})$, since $d \ln z \wedge d \ln(1 - z) = 0$. Moreover given a K_2 -symplectic structure $\omega^K = \sum_m x_m \wedge y_m$, the map $d \ln \wedge d \ln : \sum_m x_m \wedge y_m \mapsto \sum_m d \ln x_m \wedge d \ln y_m$ is a close 2-form (pre-symplectic form) on the complex variety \mathcal{M} .

Let \mathcal{M} be a complex variety with a K_2 -symplectic form $\omega^K \in K_2(\mathbb{C})$ such that $[d \ln \wedge d \ln](\omega^K) = \omega$ is a symplectic structure. A K_2 -Lagrangian subvariety $\mathcal{L}^K \in \mathcal{M}$ is a subvariety with $\dim \mathcal{L}^K = \frac{1}{2} \dim \mathcal{M}$ and

$$\omega^K|_{\mathcal{L}^K} = \sum_I z_I \wedge (1 - z_I), \quad (\text{E.3})$$

for some z_I being the holomorphic functions on \mathcal{M} . It is shown in [33, 67, 71] that at least on a generic part of $\mathcal{M}_{\text{flat}}(\Sigma_g, \text{SL}(2, \mathbb{C}))$ where we are interested in, the symplectic structure $\omega = \sum_m \frac{dx_m}{x_m} \wedge \frac{dy_m}{y_m}$ has a K_2 -avatar $\omega^K \in K_2(\mathbb{C})$ such that $[d \ln \wedge d \ln](\omega^K) = \omega$. The moduli space $\mathcal{M}_{\text{flat}}(M_3, \text{SL}(2, \mathbb{C})) = \mathcal{L}_A$ with $\partial M_3 = \Sigma_g$ is a K_2 -Lagrangian subvariety in $\mathcal{M}_{\text{flat}}(\Sigma_g, \text{SL}(2, \mathbb{C}))$, i.e. $\omega^K|_{\mathcal{L}_A} = \sum_I z_I \wedge (1 - z_I)$ for some holomorphic functions z_I .

Let's define 2 homomorphisms ϑ_k and ϑ_σ from $K_2(\mathbb{C}(\mathcal{L}_A))$ to $H^1(\mathcal{L}_A, \mathbb{R})$ (up to a $4\pi^2\mathbb{Z}$ covering for ϑ_k) by

$$\begin{aligned} \vartheta_\sigma : x \wedge y &\mapsto \vartheta_\sigma(x \wedge y) := \ln |y| d(\arg x) - \ln |x| d(\arg y) \\ \vartheta_k : x \wedge y &\mapsto \vartheta_k(x \wedge y) := \ln |y| d(\ln |x|) + \arg x d(\arg y) \end{aligned} \quad (\text{E.4})$$

Upon a choice of polarization, the Lagrangian subvariety \mathcal{L}_A is quantizable when the following conditions are satisfied for all closed path $\alpha \in \mathcal{L}_A$ (when the real part of Chern-Simons coupling $\text{Re}(t) = k \in \mathbb{Z}$) [7]:

$$\oint_\alpha \vartheta_\sigma(\omega^K|_{\mathcal{L}_A}) = 0, \quad \oint_\alpha \vartheta_k(\omega^K|_{\mathcal{L}_A}) \in 4\pi^2\mathbb{Q}. \quad (\text{E.5})$$

Since \mathcal{L}_A is a K_2 -Lagrangian subvariety with respect to ω^K , then $\vartheta_\sigma(\omega^K)$ is given by

$$\vartheta_\sigma(\omega^K|_{\mathcal{L}_A}) = \sum_I \ln |1 - z_I| d(\arg z_I) - \ln |z_I| d(\arg(1 - z_I)) = - \sum_I dD(z_I) \quad (\text{E.6})$$

where $D(z_I)$ is the Bloch-Wigner dilogarithm function

$$D(z) = \ln |z| \arg(1 - z) + \text{Im}(\text{Li}_2(z)) \quad (\text{E.7})$$

Then $\oint_{\alpha} \vartheta_{\sigma}(\omega^K|_{\mathcal{L}_A}) = 0$ is satisfied since $D(z)$ is a continuous function on \mathbb{C} . Similarly,

$$\begin{aligned}\vartheta_k(\omega^K|_{\mathcal{L}_A}) &= \sum_I \ln|1 - z_I| d(\ln|z_I|) + \arg z_I d(\arg(1 - z_I)) \\ &= - \sum_I d[\operatorname{Re}(\operatorname{Li}_2(z_I)) - \arg z_I \arg(1 - z_I)]\end{aligned}\quad (\text{E.8})$$

$\operatorname{Re}(\operatorname{Li}_2(z_I))$ is again a continuous function on \mathbb{C} , while $\sum_I \oint_{\alpha} d[\arg z_I \arg(1 - z_I)] \in 4\pi^2\mathbb{Z}$. Thus $\oint_{\alpha} \vartheta_k(\omega^K|_{\mathcal{L}_A}) \in 4\pi^2\mathbb{Q}$ indeed holds. Therefore we conclude that \mathcal{L}_A being K_2 -Lagrangian implies that \mathcal{L}_A is quantizable. In addition, the 1-forms $\vartheta_k(z \wedge (1 - z))$ and $\vartheta_{\sigma}(z \wedge (1 - z))$ are exact up to $4\pi^2\mathbb{Z}$ shows that they are indeed homomorphisms from $K_2(\mathbb{C}(\mathcal{L}_A))$ to $H^1(\mathcal{L}_A, \mathbb{R})$ up to a $4\pi^2\mathbb{Z}$ covering for ϑ_k .

When we consider the analytic continuation of Chern-Simons theory with generic non-integer k , the Lagrangian subvariety \mathcal{L}_A has to be replaced to be its cover space $\overline{\mathcal{L}}_A$, on which $\ln z_I$ is single-valued. It is because we stop quotienting large gauge transformation for analytic continued Chern-Simons theory. Then the 1-forms $\vartheta_k(z \wedge (1 - z))$ and $\vartheta_{\sigma}(z \wedge (1 - z))$ are indeed exact on the cover space $\overline{\mathcal{L}}_A$, i.e. $\oint_{\alpha} \vartheta_{\sigma} = \oint_{\alpha} \vartheta_k = 0$ on $\overline{\mathcal{L}}_A$.

F Quantization of Coadjoint Orbit, Unitary Representations of $\mathrm{SL}(2, \mathbb{C})$, and Path Integral Formula of Wilson Line

In this appendix, we give a brief review about the geometrical quantization of the coadjoint orbit in $\mathrm{SL}(2, \mathbb{C})$, which results in the unitary irreducible representations of $\mathrm{SL}(2, \mathbb{C})$. We also review briefly the path integral formula of unitary Wilson line, which is a consequence of the coadjoint orbit quantization. The reviews of the topic can be found in [85] (see also [24] for a nice summary).

As a complex Lie algebra, $\mathfrak{sl}_2\mathbb{C}$ is generated by the basis

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (\text{F.1})$$

If $\mathfrak{sl}_2\mathbb{C}$ is viewed as a real Lie algebra, it is generated by $\{E, F, H, \tilde{E} = iE, \tilde{F} = iF, \tilde{H} = iH\}$. Given $\mathfrak{sl}_2\mathbb{C}$ (viewed as a real Lie algebra) and its complexification $(\mathfrak{sl}_2\mathbb{C})_{\mathbb{C}} \simeq \mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C}$, a nondegenerate trace form $\langle \cdot, \cdot \rangle : (\mathfrak{sl}_2\mathbb{C})_{\mathbb{C}} \times \mathfrak{sl}_2\mathbb{C} \rightarrow \mathbb{C}$ is given by

$$\langle (X_L, X_R), Y \rangle = \frac{1}{2} \operatorname{tr}(X_L Y) + \frac{1}{2} \operatorname{tr}(X_R \bar{Y}), \quad (\text{F.2})$$

where $X_{L,R}, Y$ are 2×2 matrix. The trace form is a complexification of the invariant bilinear form of $\mathfrak{sl}_2\mathbb{C}$. By the trace form, a weight $\underline{\lambda} \in (\mathfrak{sl}_2\mathbb{C}^*)_{\mathbb{C}}$ can be identified as a pair of 2×2 matrices (λ_L, λ_R) in $(\mathfrak{sl}_2\mathbb{C})_{\mathbb{C}}$. The coadjoint orbit is defined by

$$(\Omega_{\lambda})_{\mathbb{C}} := \{g(\lambda_L, \lambda_R)g^{-1}\}_{g \in \mathrm{SL}(2, \mathbb{C})_{\mathbb{C}}} \simeq \mathrm{SL}(2, \mathbb{C})/\mathfrak{S}_{\lambda}^L \times \mathrm{SL}(2, \mathbb{C})/\mathfrak{S}_{\lambda}^R \quad (\text{F.3})$$

where $\mathfrak{S}_{\lambda}^{L,R}$ is the stabilizer $\mathfrak{S}_{\lambda}^{L,R} = \{h \in \mathrm{SL}(2, \mathbb{C}) | h\lambda_{L,R}h^{-1} = \lambda_{L,R}\}$. Here the stabilizer is nothing but the Cartan subgroup (or maximal torus) $\mathfrak{S}_{\lambda} = \mathbb{T}_{\mathbb{C}}$ thus the coadjoint orbit is given by

$$(\Omega_{\lambda})_{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C})/\mathbb{T}_{\mathbb{C}} \times \mathrm{SL}(2, \mathbb{C})/\mathbb{T}_{\mathbb{C}} \simeq T^*S^2 \times T^*S^2, \quad \text{with} \quad \Omega_{\lambda} = \mathrm{SL}(2, \mathbb{C})/\mathbb{T}_{\mathbb{C}} = T^*S^2 \quad (\text{F.4})$$

It is enough for us to consider the coadjoint orbit to be the real form Ω_{λ} , by viewing the second copy of T^*S^2 is the complex conjugate of the first copy.

Let $\underline{\nu}, \underline{\kappa} \in (\mathfrak{sl}_2\mathbb{C}^*)_{\mathbb{C}}$ be the linear functionals defined by $\underline{\nu}(H) = -iw$, $\underline{\kappa}(\tilde{H}) = m$ ($w, n \in \mathbb{C}$), $\underline{\nu}(\tilde{H}) = \underline{\kappa}(H) = 0$, while both $\underline{\nu}$ and $\underline{\kappa}$ annihilate $E, F, \tilde{E}, \tilde{F}$. The above trace form results in the identification $\underline{\nu} \longleftrightarrow (\nu, \nu)$ and $\underline{\kappa} \longleftrightarrow (\kappa, -\kappa)$ with ν, κ being 2×2 matrices

$$\nu = -\frac{iw}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \kappa = -\frac{im}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\text{F.5})$$

The weight $\underline{\lambda} = \underline{\nu} \oplus \underline{\kappa} \longleftrightarrow (\lambda_L, \lambda_R) = (\nu + \kappa, \nu - \kappa)$. The coadjoint orbit Ω_λ has a natural $\text{SL}(2, \mathbb{C})$ invariant symplectic structure:

$$\omega_{\nu, \kappa} = \frac{1}{2} \text{tr} \left[(\nu + \kappa) g^{-1} dg \wedge g^{-1} dg \right] + \frac{1}{2} \text{tr} \left[(\nu - \kappa) \overline{g^{-1} dg} \wedge g^{-1} dg \right]. \quad (\text{F.6})$$

By the procedure of geometrical quantization, a line-bundle $\mathfrak{Q} \rightarrow \Omega_\lambda$ must be defined over the phase space Ω_λ , while $\omega_{\nu, \kappa}$ is the curvature of \mathfrak{Q} . Due to the compact cycle $S^2 \subset \Omega_\lambda$, Weil's integrality criterion requires $\omega_{\nu, \kappa}$ to have $m \in \mathbb{Z}$ in order that \mathfrak{Q} is prequantizable. $\omega_{\nu, \kappa}$ being real implies $w \in i\mathbb{R}$. The prequantum line-bundle \mathfrak{Q} can be obtained from $\mathbb{C} \times \text{SL}(2, \mathbb{C})$ as a quotient by the representation of $\mathfrak{H}_\lambda = \mathbb{T}_\mathbb{C}$ acting on \mathbb{C} . The representation is given by $(f, x) \mapsto (\sigma(h)f, xh)$, so that the quotient is given by the identification:

$$(f, xh) = (\sigma(h^{-1})f, x) \quad \text{or} \quad f(xh) = \sigma(h^{-1})f(x), \quad \text{with } f \in \mathbb{C}, x \in \text{SL}(2, \mathbb{C}), h \in \mathbb{T}_\mathbb{C}. \quad (\text{F.7})$$

The representation $\sigma(h^{-1})$ is given by $e^{(i\nu + \underline{\rho}) \oplus i\kappa}(h^{-1})$. Here $\underline{\rho} \in \mathfrak{sl}_2 \mathbb{C}^*$ is the restricted positive root $\underline{\rho}(H) = 2$, $\underline{\rho}(\tilde{H}) = 0$ ($\underline{\rho}$ annihilates $E, F, \tilde{E}, \tilde{F}$). The above quotient gives the prequantum line-bundle $\mathfrak{Q} \rightarrow \Omega_\lambda$ where $\text{SL}(2, \mathbb{C})$ acts on the sections f by

$$g \triangleright f(x) = f(g^T x). \quad (\text{F.8})$$

An element of $\text{SL}(2, \mathbb{C})$ can be written as

$$g = \begin{pmatrix} z^1 & -x^2 \\ z^2 & x^1 \end{pmatrix} \quad \text{with} \quad z^1 x^1 + z^2 x^2 = 1 \quad (\text{F.9})$$

In the coadjoint orbit $\text{SL}(2, \mathbb{C})/\mathbb{T}_\mathbb{C}$ there is an equivalence $(z^1, z^2, x^1, x^2) \sim (\alpha z^1, \alpha z^2, \alpha^{-1} x^1, \alpha^{-1} x^2)$ for $\alpha \in \mathbb{C}^*$. We use a polarization such that the resulting sections of \mathfrak{Q} depend only on the projective coordinate z^1/z^2 . The sections transform in the following way because of the above quotient procedure Eq.(F.7)³⁴:

$$f(\alpha z^1, \alpha z^2, \bar{\alpha} \bar{z}^1, \bar{\alpha} \bar{z}^2) = \alpha^{-\frac{1}{2}(w+m)-1} \bar{\alpha}^{-\frac{1}{2}(w-m)-1} f(z^1, z^2, \bar{z}^1, \bar{z}^2), \quad \alpha \in \mathbb{C}^*. \quad (\text{F.10})$$

This transformation is precisely the scaling property of the homogeneous function/section in the principle series representation when $w \in i\mathbb{R}$ and $m \in \mathbb{Z}$. In our analysis of knotted graph operator, the parameters w, m are given by

$$w = -2i\gamma j_{ab}, \quad m = -2j_{ab}, \quad j_{ab} \in \mathbb{Z}/2. \quad (\text{F.11})$$

The group action Eq.(F.8) gives the representation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \triangleright f(z, \bar{z}) = (bz + d)^{-\frac{1}{2}(w+m)-1} (\bar{b}\bar{z} + \bar{d})^{-\frac{1}{2}(w-m)-1} f\left(\frac{az + c}{bz + d}\right), \quad \text{where } z = \frac{z^1}{z^2} \quad (\text{F.12})$$

z is the projective coordinate on \mathbb{CP}^1 . The space of above sections on \mathbb{CP}^1 completed by the L^2 inner product with measure $dz = \frac{i}{2}(z^1 dz^2 - z^2 dz^1) \wedge (\bar{z}^1 d\bar{z}^2 - \bar{z}^2 d\bar{z}^1)$ carries the principle series unitary irreducible representation of $\text{SL}(2, \mathbb{C})$ labeled by (m, w) . The carrier space is denoted by $\mathcal{H}^{m, w}$ or $\mathcal{H}^{j, \rho}$ with $m = -2j$ and $w = -2i\rho$. There is an isomorphism between the representations related by (m, w) and $(-m, -w)$.

In the above representation using sections on \mathbb{CP}^1 , the variables z^1, z^2 are the ‘‘position variables’’ which corresponds to the multiplication operator on (a dense domain of) $\mathcal{H}^{m, w}$. The variable x^1, x^2 are ‘‘momentum variables’’, which corresponds to the derivative operators:

$$x^1 = \left(\frac{2}{w+m} \right) \frac{\partial}{\partial z^1}, \quad x^2 = \left(\frac{2}{w+m} \right) \frac{\partial}{\partial z^2}, \quad \bar{x}^1 = \left(\frac{2}{w-m} \right) \frac{\partial}{\partial \bar{z}^1}, \quad \bar{x}^2 = \left(\frac{2}{w-m} \right) \frac{\partial}{\partial \bar{z}^2}. \quad (\text{F.13})$$

³⁴Let $h = e^{tH}$ with $t \in \mathbb{C}$, we have $[(i\nu + \underline{\rho}) \oplus i\kappa](tH) = \frac{1}{2} \text{tr}[(i\nu + i\kappa + \rho)tH] + \frac{1}{2} \text{tr}[(i\nu - i\kappa + \rho)\bar{t}H] = \frac{t}{2}(w + k + 2) + \frac{\bar{t}}{2}(w - k + 2)$, where the 2×2 matrix ρ equals ν when $w = 2i$.

The scaling property Eq.(F.10) implies

$$z^1 \frac{\partial}{\partial z^1} + z^2 \frac{\partial}{\partial z^2} = -\frac{1}{2}(w+m) - 1, \quad \bar{z}^1 \frac{\partial}{\partial \bar{z}^1} + \bar{z}^2 \frac{\partial}{\partial \bar{z}^2} = -\frac{1}{2}(w-m) - 1. \quad (\text{F.14})$$

We remark that the unitary irrep constructed above is the *induce representation* $\text{ind}_B^{\text{SL}(2, \mathbb{C})}(\sigma)$ on the sections of a line-bundle over the coset $\text{SL}(2, \mathbb{C})/B \simeq \mathbb{CP}^1$. B is the Borel subgroup of upper-triangular matrices, whose Lie algebra is generated by $H, \tilde{H}, E, \tilde{E}$. The sections are obtained from the functions f on $\text{SL}(2, \mathbb{C})$ satisfying

$$f(xb) = \sigma(b^{-1})f(x), \quad \text{where } b \in B, x \in \text{SL}(2, \mathbb{C}). \quad (\text{F.15})$$

σ is given by $\sigma = e^{(i\nu+\rho)\oplus i\kappa}$ viewed as a representation of B .

The Wilson line in the unitary irrep (m, w) can be written as a path integral, when we consider its matrix element in the z -space representation (z is the projective coordinate of \mathbb{CP}^1):

$$\langle z | \mathcal{P} e^{\int_\ell A} | z' \rangle_{\mathcal{H}_{m,w}} = \int_{z'}^z \mathcal{D}g \mathcal{D}\bar{g} e^{iS[g, \bar{g}; A, \bar{A}]}, \quad (\text{F.16})$$

where the Lagrangian $S[g, \bar{g}; A, \bar{A}]$ is given by:

$$S[g, \bar{g}; A, \bar{A}] = -\frac{1}{2} \int_\ell \text{tr} \left[(\nu + \kappa) g^{-1} (d + A^T) g + (\nu - \kappa) \bar{g}^{-1} (d + \bar{A}^T) \bar{g} \right] \quad (\text{F.17})$$

The path integral has a first-order Lagrangian depending on the $\text{SL}(2, \mathbb{C})$ -valued functions $g : \ell \rightarrow \text{SL}(2, \mathbb{C})$. The boundary condition for the path integral is that the “position variables” of g at the source and target of ℓ equal to z' and z . The above path integral can be viewed as a quantum particle moving on its “position space” \mathbb{CP}^1 .

However there is a gauge symmetry of the Lagrangian, i.e. $S[g, \bar{g}; A, \bar{A}]$ is invariant under $g \mapsto gh$ with $h \in \mathfrak{H}_\lambda = \mathbb{T}_\mathbb{C}$ when h is trivial on the boundary. Therefore the path integral is essentially defined to be over the maps $g : \ell \rightarrow \text{SL}(2, \mathbb{C})/\mathbb{T}_\mathbb{C} = \Omega_\lambda$, where Ω_λ is the coadjoint orbit (the phase space), except the integral at the boundary of ℓ . If we consider the gauge transformation $g \mapsto gh$ with $h \in \mathfrak{H}_\lambda = \mathbb{T}_\mathbb{C}$ being non-trivial on the boundary, the path integral Eq.(F.16) transforms non-trivially. The result of path integral defines a section in a line-bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$. Indeed, let's consider an arbitrary gauge transformation $g \mapsto gh$ with $h = e^{\tau H}$, $\tau \in \mathbb{C}$. The action S transforms as

$$S[g, \bar{g}; A, \bar{A}] \mapsto S[g, \bar{g}; A, \bar{A}] + \frac{i}{2}(w+m) \int_\ell d\tau + \frac{i}{2}(w-m) \int_\ell d\bar{\tau}. \quad (\text{F.18})$$

Since the transformation gives $g \mapsto gh$ the scaling of the coordinates z^1, z^2, x^1, x^2 for the quotient $\text{SL}(2, \mathbb{C})/\mathbb{T}_\mathbb{C}$

$$\begin{pmatrix} z^1 & -x^2 \\ z^2 & x^1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} \alpha z^1 & -\alpha^{-1} x^2 \\ \alpha z^2 & \alpha^{-1} x^1 \end{pmatrix}, \quad \text{where } \alpha = e^\tau. \quad (\text{F.19})$$

Eq.(F.18) implies the path integral transform in the same way as Eq.(F.10), which has to be the case in order that Eq.(F.16) is correct and the L^2 inner product with dz is scaing invariant, i.e.

$$\begin{aligned} \int_{z'}^{z^2} \mathcal{D}g \mathcal{D}\bar{g} e^{iS[g, \bar{g}; A, \bar{A}]} &= \alpha^{-\frac{1}{2}(w+m)-1} \bar{\alpha}^{-\frac{1}{2}(w-m)-1} \int_{z'}^z \mathcal{D}g \mathcal{D}\bar{g} e^{iS[g, \bar{g}; A, \bar{A}]} \\ \int_{\lambda z'}^z \mathcal{D}g \mathcal{D}\bar{g} e^{iS[g, \bar{g}; A, \bar{A}]} &= \alpha^{\frac{1}{2}(w+m)-1} \bar{\alpha}^{\frac{1}{2}(w-m)-1} \int_{z'}^z \mathcal{D}g \mathcal{D}\bar{g} e^{iS[g, \bar{g}; A, \bar{A}]} \end{aligned} \quad (\text{F.20})$$

Note that there is a scaling factor $\alpha^{-1} \bar{\alpha}^{-1}$ in the above coming from the path integral measure at the boundary.

With the boundary condition on z, z' , the equation of motion can be derived from S by the variational principle:

$$\left[\nu + \kappa, g^{-1} (d + A^T) g \right] = 0, \quad \left[\nu - \kappa, \bar{g}^{-1} (d + \bar{A}^T) \bar{g} \right] = 0 \quad (\text{F.21})$$

which implies that g is the gauge transformation diagonalizing the component of A_t along the curve ℓ , or namely,

$$\frac{d}{dt}g + A_t^T g \propto_{\mathbb{C}} gH, \quad \frac{d}{dt}\bar{g} + \bar{A}_t^T \bar{g} \propto_{\mathbb{C}} \bar{g}H. \quad (\text{F.22})$$

Again by the expression of g by using the coordinates z^1, z^2, x^1, x^2 , we obtain that $\frac{d}{dt}z + A_t^T z \propto_{\mathbb{C}} z$ (similar for \bar{z}) where $z = (z^1, z^2)^T$. Then we obtain the on-shell relation for the boundary data z, z' of path integral:

$$z \propto_{\mathbb{C}} \mathcal{P}e^{-\int_{\ell} A^T} z'. \quad (\text{F.23})$$

The Hamiltonian analysis of $S[g]$ reproduces the symplectic structure $\omega_{v,\kappa}$ in Eq.(F.6), and gives the Hamiltonian $\mathbf{H} = p \cdot \partial_t q - \mathbf{L}$

$$\mathbf{H} = \frac{1}{2} \text{tr} \left[(\nu + \kappa) g^{-1} A_t^T g + (\nu - \kappa) \bar{g}^{-1} \bar{A}_t^T \bar{g} \right] \quad (\text{F.24})$$

where A_t is the component of A along the curve ℓ . We replace the variables in g by the corresponding operators in the z -space representation to define the Hamiltonian operator $\hat{\mathbf{H}}$. Here A_t is treated as the external variable, so its components in $\mathfrak{sl}_2\mathbb{C}$ basis are treated as c-numbers. Then it can be shown that the resulting Hamiltonian operator $-i\hat{\mathbf{H}}$ is nothing but the representation of $A_t : \ell \rightarrow \mathfrak{sl}_2\mathbb{C}$ in the unitary irrep as an operator on $\mathcal{H}^{m,w}$, namely if we expand the 2 matrix $A_t = aH + bE + cF$, then

$$-i\hat{\mathbf{H}} = a\hat{H} + b\hat{E} + c\hat{F} \quad (\text{F.25})$$

where $\hat{H}, \hat{E}, \hat{F}$ are the differential operators representing $H, E, F \in \mathfrak{sl}_2\mathbb{C}$ and generating infinitesimally the representation Eq.(F.12). As a result, the path integral form Eq.(F.16) of Wilson line is a consequence of the following relation from quantum mechanics:

$$\langle z | T e^{-i \int_{\ell} \hat{\mathbf{H}} dt} | z' \rangle = \int_{\mathcal{Z}} \mathcal{D}g \mathcal{D}\bar{g} e^{iS[g, \bar{g}; A, \bar{A}]}. \quad (\text{F.26})$$

T denotes the time-ordering which corresponds to the path ordering \mathcal{P} for the Wilson line.

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